Artin-Schreier extensions of minimal genus

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The problem

- $\mathcal{X} :=$ (projective, non-singular, geometrically irreducible) algebraic curve of genus $g(\mathcal{X}) \geq 2$, defined over an algebraically closed filed $\mathbb{K}$ of characteristic $p > 0$, with function field $\mathbb{K}(\mathcal{X})$.

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- GOAL: To construct curves with many automorphisms.

A strategy

We consider the following general problem:

Let $\mathcal{Y} : f(x, y) = 0$ be a curve of genus $g(\mathcal{Y}) \geq 2$ and p-rank $\gamma(\mathcal{Y}) = 0$. Characterize functions $u \in \mathbb{K}(\mathcal{Y})$ for which the ramified Artin-Schreier cover

$$\mathcal{X} : \begin{cases} z^p - z = u \\ f(x, y) = 0 \end{cases}$$

has the smallest possible genus, that is, $g(\mathcal{X}) = p \cdot g(\mathcal{Y})$.

The minimal genus

Let $g(\mathcal{Y})$ be the genus of the quotient curve $\mathcal{Y} = \mathcal{X}/G$. The Hurwitz genus formula the following equation gives

$$2g(\mathcal{X}) - 2 = |G|(2g(\mathcal{Y}) - 2) + \sum_{P \in \mathcal{X}} d_P, \tag{1}$$

where

$$d_P = \sum_{i \geq 0} \left(\left| G_P^{(i)} \right| - 1 \right).$$

In particular, if G is a p-group, then $G_P^{(0)} = G_P^{(1)}$, and then

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- if $|G| = p$, then $|G_P^{(0)|} = |G_P^{(1)}| = p$ gives $d_P \geq 2(p-1)$, and then (1) gives

$$g(\mathcal{X}) \geq p \cdot g(\mathcal{Y}).$$

Back to Aut($\mathcal{X}$): a few facts

Let $\mathcal{X}$ be a curve of genus $g(\mathcal{X}) \geq 2$ over an algebraically closed filed $\mathbb{K}$ of characteristic $p \geq 0$, with function field $\mathbb{K}(\mathcal{X})$. By definition,

$$\mathrm{Aut}(\mathcal{X}) = \mathrm{Aut}_{\mathbb{K}}(\mathbb{K}(\mathcal{X})).$$

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- Stichtenoth (1973): $|\mathrm{Aut}(\mathcal{X})| < 16g^4$, unless $\mathcal{X}$ is the Hermitian curve.
- Henn (1976): $|\mathrm{Aut}(\mathcal{X})| < 8g^3$, up to four exceptions: A certain hyperelliptic curve over $\overline{\mathbb{F}}_2$, the Roquette curve, the Hermitian curve, and the Suzuki curve.

Two more families of curves $\mathcal{X}$ with large $\mathrm{Aut}(\mathcal{X})$

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- **common property**: They all have ZERO p-rank.

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Other examples of curves with large automorphism groups are the **Ree**, and the **GK** curves. All such curves have the following:

- **common property**: They all have ZERO p-rank.
- That is, they are curves with no unramified p-cyclic covers.

Zero p-rank vs. Large automorphism groups

- (Nakajima (1987)) Let $\mathcal{X}$ be a curve with $g := g(\mathcal{X}) \geq 2$ and S a Sylow p-subgroup of $\mathrm{Aut}(\mathcal{X})$. If $\gamma = \gamma(\mathcal{X})$ is the p-rank of $\mathcal{X}$, then

$$|S| \leq \begin{cases} 4(g-1) & \text{if } \gamma > 0 \\ \max\left\{g, 4pg^2/(p-1)^2\right\} & \text{if } \gamma = 0 \end{cases}$$

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- (Giulietti -Korchmáros (2019)) For $p > 2$ and $G \leq \mathrm{Aut}(\mathcal{X})$, there exists a constant $c_p > 0$ such that if $|G| > c_p g^2$, then the p-rank of $\mathcal{X}$ is zero.

p-ranks in p-power Galois covers

Theorem (Deuring-Shafarevich)

Let $G \leq \mathrm{Aut}(\mathcal{X})$ be such that $|G| = p^r$. Then

$$\gamma - 1 = p^r \left(\gamma' - 1\right) + \sum_{i=1}^{s} \left(p^r - \ell_i\right),$$

where γ and γ' are the p-ranks of $\mathcal{X}$ and its quotient curve $\mathcal{X}/G$, while $\ell_1, \ldots, \ell_s$ are the sizes of the short orbits of G on the points of $\mathcal{X}$.

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Corollary

Let $\mathcal{Y} : f(x, y) = 0$ be a curve with $\gamma(\mathcal{Y}) = 0$. If $\pi : \mathcal{X} \longrightarrow \mathcal{Y}$ is an Artin-Schreier extension of minimal genus, then $\gamma(\mathcal{X}) = 0$.

Back to our problem:

Construct curves $\mathcal{X}$ with many automorphisms arising from:

Let $\mathcal{Y} : f(x, y) = 0$ be a curve with many automorphisms (and zero p-rank), and let $u \in \mathbb{K}(\mathcal{Y})$ be a function for which the ramified Artin-Schreier cover

$$\mathcal{X} : \begin{cases} z^p - z = u \\ f(x, y) = 0 \end{cases}$$

has the smallest possible genus, namely $g(\mathcal{X}) = p \cdot g(\mathcal{Y})$.

A prototype

Theorem

Up to isomorphism,

$$\mathcal{X} : y^q + y = (x^p - x)^{q+1}$$

is the only Artin-Schreier cover of minimal genus of the Hermitian curve $\mathcal{H} : y^q + y = x^{q+1}$*, ramified over* $Q_\infty \in \mathcal{H}$*.*

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Proof.

Artin-Schreier theory. □

A curve with many automorphisms

The curve

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- LOW genus $g(\mathcal{X}) = pq(q-1)/2$.

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- LOW genus $g(\mathcal{X}) = pq(q-1)/2$.
- MANY rational points
- ZERO p-rank
- A **LARGE** automorphism group.

The automorphism group of $\mathcal{X}$

Theorem

Let $\mathcal{X} : y^q + y = (x^p - x)^{q+1}$, *with* $p < q$. *Then* $\mathrm{Aut}(\mathcal{X})$ *is given by the set of maps*

$$\varphi_{a,b,c} : (x, y) \mapsto (ax + b, a^{q+1}y + a(b^p - b)^q(x^p - x) + c),$$

where $a \in \mathbb{F}_q^*$, $c \in \mathbb{F}_{q^2}$, *and* $b \in \mathbb{K}$ *such that* $c^q + c = (b^p - b)^{q+1}$. *In particular,* $|\mathrm{Aut}(\mathcal{X})| = pq^3(p-1) > 4g(\mathcal{X})^{3/2}$.

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- Checking that $\varphi_{a,b,c} \in \mathrm{Aut}(\mathcal{X})$ is straightforward. Since all such maps fix the point $P_\infty \in \mathcal{X}$, we have $pq^3(p-1) | \#\,\mathrm{Aut}(\mathcal{X})_{P_\infty}$, and then the unique Sylow p-subgroup of $\mathrm{Aut}(\mathcal{X})_{P_\infty}$ satisfies

$$\mathrm{Aut}(\mathcal{X})_{P_\infty}^{(1)} \geq pq^3 > pq(q-1) + 1 = 2g(\mathcal{X}) + 1.$$

A special ingredient for the proof

Theorem (Giuiletti-Korchmáros (2007))

Let $\mathcal{X}$ be a curve of genus $g \geq 2$ over $\mathbb{F}_q$, and let $\mathbb{G} \leq \mathrm{Aut}(\mathcal{X})$ be such that $|\mathbb{G}_P^{(1)}| > 2g+1$ for some point $P \in \mathcal{X}$. Then one of the following cases occurs:

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   - *the Hermitian curve.*
   - *the Suzuki curve $\mathbf{v}(X^{n_0}(X^n + X) - (Y^n + Y))$ with $p = 2$, $q = n$, $n_0 = 2^r$, $r \geq 1$, $n = 2n_0^2$ and $g = n_0(n-1)$.*

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   - *the Ree curve* $\mathbf{v}(Y^{n^2} - (1 + (X^n - X)^{n-1})Y^n + (X^n - X)^{n-1}Y - X^n(X^n - X)^{n+3n_0}$ *with* $p = 3$, $q = n$, $n_0 = 3^r$, $n = 3n_0^2$ *and* $g = \frac{3}{2}n_0(n-1)(n+n_0+1)$.

The Weierstrass semigroup at P_∞

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Let $\mathcal{X}: y^q + y = (x^p - x)^{q+1}$, with $p < q$. Then

1. $H(P_\infty) = \langle q, p(q+1), q^2+1 \rangle$, *and*
2. $\mathcal{L}(qP_\infty) = \langle 1, x \rangle$ *and* $\mathcal{L}(p(q+1)P_\infty) = \langle 1, x, \ldots, x^p, y \rangle$.

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- Note that $\mathrm{Aut}(\mathcal{X}) = \mathrm{Aut}(\mathcal{X})_{P_\infty}$ implies that the Riemann-Roch spaces $\mathcal{L}(qP_\infty)$ and $\mathcal{L}(p(q+1)P_\infty)$ are invariant by $\mathrm{Aut}(\mathcal{X})$. Thus any $\sigma \in \mathrm{Aut}(\mathcal{X})$ is such that

$$\sigma(x) = ax + b \quad \text{and } \sigma(y) = cy + f(x),$$

where $\deg f(x) \leq p$.

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  where $\deg f(x) \leq p$.
- Taking the equation of $\mathcal{X}$ into account, the result follows.

Elementary abelian p-extensions

- Via iteration, one can easily check that the "minimal genus condition" arising from the inequality $g(\mathcal{X}) \geq p \cdot g(\mathcal{Y})$, in the Artin-Schreier setting, can be extended to the case of elementary abelian p-extensions of degree p^r, where we have

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$$g(\mathcal{X}) \geq p^r \cdot g(\mathcal{Y}).$$
- In particular, we may consider curves of type
$$\mathcal{X} : \ y^q + y = L(x)^{q+1},$$
where $L(x)$ is an additive separable polynomial.

For example: $y^q + y = (x^{p^r} - x)^{q+1}$

Using the same method, one can study the automorphism group of $\mathcal{X}_r : y^q + y = (x^{p^r} - x)^{q+1}$ and prove that if $p^r \geq q$, then

$$|\operatorname{Aut} \mathcal{X}_r| = p^{2r} (p^r - 1) q^3.$$

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In particular, since $g(\mathcal{X}_r) = p^r q(q-1)/2$, we have

- $|\operatorname{Aut} \mathcal{X}_r| > 4g(\mathcal{X}_r)^2$ (a very large automorphism group)

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- Furthermore, note that the case $q = p$ gives

$$|\operatorname{Aut} \mathcal{X}_r| = p^{2r+3} (p^r - 1),$$

and this provides a family of curves $\mathcal{X}_r$ for which the p-Sylow subgroups $S \leq \operatorname{Aut}(\mathcal{X}_r)$ attain Nakajima's bound

$$|S| \leq \max\left\{g, 4pg^2/(p-1)^2\right\} = p^{2r+3}.$$

A remark

- Note that there is already an extensive study on the automorphism groups of curves of type

$$\mathcal{C} = \mathbf{v}(A(Y) - B(X))$$

where $A(Y)$ is linearized polynomial, and $p \nmid \deg B$.

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$$\mathcal{C} = \mathbf{v}(A(Y) - B(X))$$

  where $A(Y)$ is linearized polynomial, and $p \nmid \deg B$.
- In general, for the cases considered here, we have $p \mid \deg B$. For example, $y^q + y = (x^p - x)^{q+1}$.

The End

THANK YOU!