On the number of rational points of curves over a surface in $\mathbb{P}^3$

Elena Berardini[1] & Jade Nardi[2]

1. Eindhoven University of Technology 2. IRMAR, CNRS, Univ Rennes 1

EuroTechPostdoc2 Programme
Excellence in Science and Technology

14^{th} February 2023
Conference **O**n al**G**ebraic varieties over fi**N**ite fields and **A**lgebraic geometry **C**odes

Acta Arithmetica 205 (2022)
`https://arxiv.org/abs/2111.09578`

Today is February 14^{th}! Looking for the perfect gift for Valentine's Day?

Today is February 14^{th}! Looking for the perfect gift for Valentine's Day?

Today is February 14^{th}! Looking for the perfect gift for Valentine's Day?

Today is February 14^{th}! Looking for the perfect gift for Valentine's Day?

Today is February 14^{th}! Looking for the perfect gift for Valentine's Day?

Offer a theorem instead:

THEOREMS ARE FOREVER[1]!

1. See the end of the talk for my endless gift!

Curves, surfaces, rational points and all that jazz

We let $\mathbb{F}_q$ denote a finite field with q elements and $\mathbb{P}^n_{\mathbb{F}_q}$ the projective space.

An algebraic projective variety X defined over $\mathbb{F}_q$ is the set of zeros of homogenous polynomials $f_1, \dots, f_r \in \mathbb{F}_q[x_0, \dots, x_n]$ irreducible over $\mathbb{F}_q$:

$$X \overset{\text{def}}{=} \{P \in \mathbb{P}^n \mid f_1(P) = \cdots = f_r(P) = 0\}.$$

Curves, surfaces, rational points and all that jazz

We let $\mathbb{F}_q$ denote a finite field with q elements and $\mathbb{P}^n_{\mathbb{F}_q}$ the projective space.

An algebraic projective variety X defined over $\mathbb{F}_q$ is the set of zeros of homogenous polynomials $f_1, \ldots, f_r \in \mathbb{F}_q[x_0, \ldots, x_n]$ irreducible over $\mathbb{F}_q$:

$$X \stackrel{\text{def}}{=} \{P \in \mathbb{P}^n \mid f_1(P) = \cdots = f_r(P) = 0\}.$$

The set of rational points of X is

$$X(\mathbb{F}_q) \stackrel{\text{def}}{=} \{P = (a_0 : \cdots : a_n) \in X \mid \forall\, i,\ a_i \in \mathbb{F}_q\}$$

Curves, surfaces, rational points and all that jazz

We let $\mathbb{F}_q$ denote a finite field with q elements and $\mathbb{P}^n_{\mathbb{F}_q}$ the projective space.

An algebraic projective variety X defined over $\mathbb{F}_q$ is the set of zeros of homogenous polynomials $f_1, \ldots, f_r \in \mathbb{F}_q[x_0, \ldots, x_n]$ irreducible over $\mathbb{F}_q$:

$$X \overset{\text{def}}{=} \{P \in \mathbb{P}^n \mid f_1(P) = \cdots = f_r(P) = 0\}.$$

The set of rational points of X is

$$X(\mathbb{F}_q) \overset{\text{def}}{=} \{P = (a_0 : \cdots : a_n) \in X \mid \forall\, i,\ a_i \in \mathbb{F}_q\} = \{P \in X \mid \overset{\text{Frobenius morphism}}{\Phi}(P) = P\}.$$

Curves, surfaces, rational points and all that jazz

We let $\mathbb{F}_q$ denote a finite field with q elements and $\mathbb{P}^n_{\mathbb{F}_q}$ the projective space.

An algebraic projective variety X defined over $\mathbb{F}_q$ is the set of zeros of homogenous polynomials $f_1, \ldots, f_r \in \mathbb{F}_q[x_0, \ldots, x_n]$ irreducible over $\mathbb{F}_q$:

$$X \overset{\text{def}}{=} \{P \in \mathbb{P}^n \mid f_1(P) = \cdots = f_r(P) = 0\}.$$

The set of rational points of X is

$$X(\mathbb{F}_q) \overset{\text{def}}{=} \{P = (a_0 : \cdots : a_n) \in X \mid \forall\, i,\ a_i \in \mathbb{F}_q\} = \{P \in X \mid \overset{\text{Frobenius morphism}}{\Phi}(P) = P\}.$$

Today: algebraic varieties of dimension one (curves C) and two (surfaces S) in $\mathbb{P}^3$.

Existing bounds

Theorem [Hasse–Weil, 1948]

If C is an absolutely irreducible smooth curve of genus g defined over the finite field $\mathbb{F}_q$, then $\#C(\mathbb{F}_q) \leq q + 1 + 2g\sqrt{q}$.

Existing bounds

Theorem [Hasse–Weil, 1948, Aubry–Perret, 1993]

If C is an absolutely irreducible ~~smooth~~ curve of arithmetic genus π defined over the finite field $\mathbb{F}_q$, then $\#C(\mathbb{F}_q) \leq q + 1 + 2\pi\sqrt{q}$.

Existing bounds

Theorem [Hasse–Weil, 1948, Aubry–Perret, 1993]

If C is an absolutely irreducible curve of arithmetic genus π defined over the finite field $\mathbb{F}_q$, then $\#C(\mathbb{F}_q) \leq q + 1 + 2\pi\sqrt{q}$.

Theorem [Homma, 2012]

If C is a non–degenerate curve defined over $\mathbb{F}_q$ of degree δ in $\mathbb{P}^n$, with $n \geq 3$, then $\#C(\mathbb{F}_q) \leq (\delta - 1)q + 1$.

Existing bounds

Theorem [Hasse–Weil, 1948, Aubry–Perret, 1993]

If C is an absolutely irreducible curve of arithmetic genus π defined over the finite field $\mathbb{F}_q$, then $\#C(\mathbb{F}_q) \leq q + 1 + 2\pi\sqrt{q}$.

Theorem [Homma, 2012]

If C is a non–degenerate curve defined over $\mathbb{F}_q$ of degree δ in $\mathbb{P}^n$, with $n \geq 3$, then $\#C(\mathbb{F}_q) \leq (\delta - 1)q + 1$.

Theorem [Stöhr–Voloch, 1986]

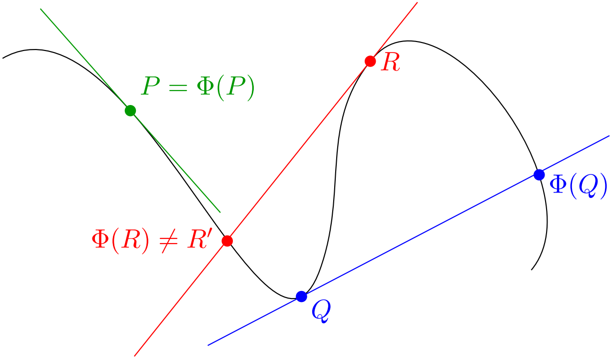
Let $C/\mathbb{F}_q$ be an irreducible smooth curve of genus g and degree δ in $\mathbb{P}^n$. Let $\nu_1, \ldots, \nu_{n-1}$ be its Frobenius orders (generically $\nu_i = i$). Then

$$\#C(\mathbb{F}_q) \leq \frac{1}{n}\left((\nu_1 + \cdots + \nu_{n-1})(2g - 2) + (q + n)\delta\right).$$

Stöhr and Voloch's strategy for plane curves

Take C a plane curve of deg. δ defined by $f = 0$ over $\mathbb{F}_q$. Write Φ for the q–Frobenius morphism.

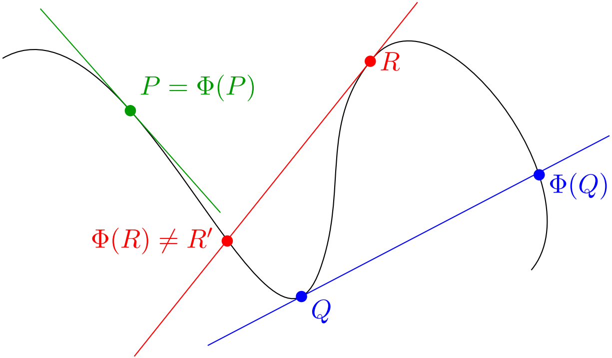
$$\begin{array}{rcl} C(\mathbb{F}_q) & = & \{P \in C \mid \Phi(P) = P\} \\ & & \qquad\quad \cap\!\!| \\ & & \{P \in C \mid \Phi(P) \in T_P C\} \stackrel{\text{def}}{=} \mathcal{Z}. \end{array}$$



Stöhr and Voloch's strategy for plane curves

Take C a plane curve of deg. δ defined by $f=0$ over $\mathbb{F}_q$. Write Φ for the q–Frobenius morphism.

$$\begin{array}{rcl} C(\mathbb{F}_q) & = & \{P \in C \mid \Phi(P) = P\} \\ & & \quad\quad\quad \cap\!\!| \\ & & \{P \in C \mid \Phi(P) \in T_P C\} \stackrel{\text{def}}{=} \mathcal{Z}. \end{array}$$

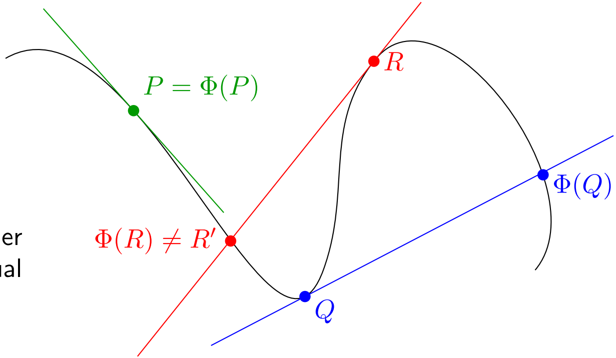


Set $g(X,Y) = X^q f_X + Y^q f_Y + Z^q f_Z$.
Then $\mathcal{Z} = C \cap (g=0)$.

Stöhr and Voloch's strategy for plane curves

Take C a plane curve of deg. δ defined by $f = 0$ over $\mathbb{F}_q$. Write Φ for the q–Frobenius morphism.

$$\begin{aligned} C(\mathbb{F}_q) &= \{P \in C \mid \Phi(P) = P\} \\ &\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \cap\!\!\!| \\ & \{P \in C \mid \Phi(P) \in T_P C\} \overset{\text{def}}{=} \mathcal{Z}. \end{aligned}$$



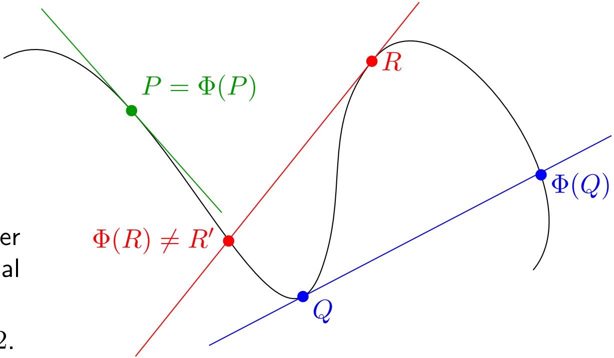
Set $g(X,Y) = X^q f_X + Y^q f_Y + Z^q f_Z$.
Then $\mathcal{Z} = C \cap (g = 0)$.

Bézout's theorem: if $\dim \mathcal{Z} = 0$, the number of points in $\mathcal{Z}$ counted with *multiplicity* is equal to $(\deg f) \cdot (\deg g) = \delta(\delta + q - 1)$.

Stöhr and Voloch's strategy for plane curves

Take C a plane curve of deg. δ defined by $f=0$ over $\mathbb{F}_q$. Write Φ for the q–Frobenius morphism.

$$\begin{aligned} C(\mathbb{F}_q) \quad &= \quad \{P \in C \mid \Phi(P) = P\} \\ &\qquad\qquad\qquad \cap\!\!| \\ &\quad \{P \in C \mid \Phi(P) \in T_P C\} \stackrel{\text{def}}{=} \mathcal{Z}. \end{aligned}$$



Set $g(X,Y) = X^q f_X + Y^q f_Y + Z^q f_Z$.
Then $\mathcal{Z} = C \cap (g=0)$.

Bézout's theorem: if $\dim \mathcal{Z} = 0$, the number of points in $\mathcal{Z}$ counted with *multiplicity* is equal to $(\deg f)\cdot(\deg g) = \delta(\delta+q-1)$.

Multiplicity: If $P \in C(\mathbb{F}_q)$, then $m_P(\mathcal{Z}) \geq 2$.

Stöhr and Voloch's strategy for plane curves

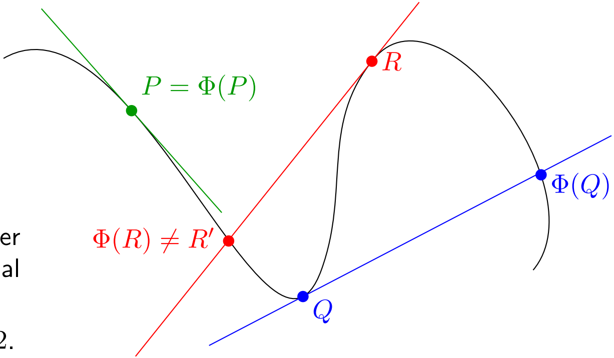
Take C a plane curve of deg. δ defined by $f = 0$ over $\mathbb{F}_q$. Write Φ for the q–Frobenius morphism.

$$C(\mathbb{F}_q) = \{P \in C \mid \Phi(P) = P\} \subseteq \{P \in C \mid \Phi(P) \in T_P C\} \overset{\text{def}}{=} \mathcal{Z}.$$

Set $g(X,Y) = X^q f_X + Y^q f_Y + Z^q f_Z$.
Then $\mathcal{Z} = C \cap (g = 0)$.

Bézout's theorem: if $\dim \mathcal{Z} = 0$, the number of points in $\mathcal{Z}$ counted with *multiplicity* is equal to $(\deg f) \cdot (\deg g) = \delta(\delta + q - 1)$.

Multiplicity: If $P \in C(\mathbb{F}_q)$, then $m_P(\mathcal{Z}) \geq 2$.



Theorem [Stöhr–Voloch, 1986]

If C has at least a non–flex point $(\Rightarrow \dim \mathcal{Z} = 0)$, then $\#C(\mathbb{F}_q) \leq \frac{1}{2}\delta(\delta + q - 1)$.

Ideas & Motivations

Let $C \subset S \hookrightarrow \mathbb{P}^n$ (via a very ample divisor).

Goal: bounding $\#C(\mathbb{F}_q)$ in terms of the embedding.

(features of the surface S and the ambient $\mathbb{P}^n$)

Main motivations:

- New bound for the number of rational points on projective curves.

  (hopefully improving the previous ones)

- Application to geometric coding theory.

Ideas & Motivations

Let $C \subset S \hookrightarrow \mathbb{P}^n$ (via a very ample divisor).

Goal: bounding $\#C(\mathbb{F}_q)$ in terms of the embedding.

(features of the surface S and the ambient $\mathbb{P}^n$)

Main motivations:

- New bound for the number of rational points on projective curves.

  (hopefully improving the previous ones)

- Application to geometric coding theory.

Bounding the minimum distance of a code from a surface S	$\rightsquigarrow$	Bounding $\#C(\mathbb{F}_q)$ for the irreducible curves C on S
Better lower bound for the minimum distance	$\Longleftrightarrow$	Better upper bound for $\#C(\mathbb{F}_q)$

Strategy ($n = 3$)

Let $S : (f = 0) \subset \mathbb{P}^3$ be a smooth irreducible algebraic surface of degree d defined $\mathbb{F}_q$.

Set $C_\Phi^S \overset{\text{def}}{=} \{P \in S \mid \Phi(P) \in T_P S\}$. Then $S(\mathbb{F}_q) \subset C_\Phi^S$.

Strategy ($n = 3$)

Let $S : (f = 0) \subset \mathbb{P}^3$ be a smooth irreducible algebraic surface of degree d defined $\mathbb{F}_q$.

Set $C_\Phi^S \overset{\text{def}}{=} \{P \in S \mid \Phi(P) \in T_P S\}$. Then $S(\mathbb{F}_q) \subset C_\Phi^S$.

$$C_\Phi^S : f = h = 0 \text{ for } h := X_0^q f_0 + X_1^q f_1 + X_2^q f_2 + X_3^q f_3 \Rightarrow \deg h = d + q - 1.$$

Strategy $(n = 3)$

Let $S : (f = 0) \subset \mathbb{P}^3$ be a smooth irreducible algebraic surface of degree d defined $\mathbb{F}_q$.

Set $C_\Phi^S \overset{\text{def}}{=} \{P \in S \mid \Phi(P) \in T_P S\}$. Then $S(\mathbb{F}_q) \subset C_\Phi^S$.

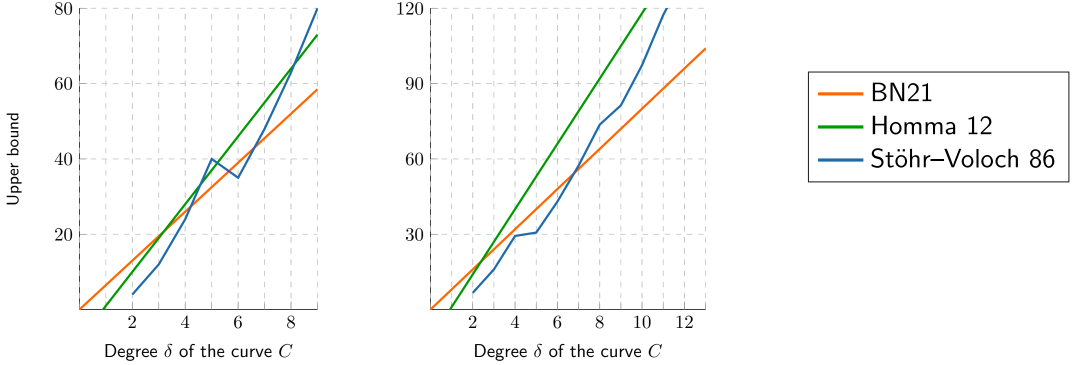
$$C_\Phi^S : f = h = 0 \text{ for } h := X_0^q f_0 + X_1^q f_1 + X_2^q f_2 + X_3^q f_3 \Rightarrow \deg h = d + q - 1.$$

Take a curve $C \subset S$ of degree δ. Then $C(\mathbb{F}_q) \subseteq C \cap C_\Phi^S$.

If $C \cap C_\Phi^S$ is a finite set of points, then

$$\#C(\mathbb{F}_q) \leq \frac{\deg(C \cap C_\Phi^S)}{\min_{P \in C(\mathbb{F}_q)} m_P(C, C_\Phi^S)} \leq \frac{\delta(d + q - 1)}{2}.$$

Comparisons with pre–existing bounds



(a) $q = 9$ and $d = 5$

(b) $q = 13$ and $d = 4$

Figure: Bounds on the number of $\mathbb{F}_q$–points on a non–plane curve C on a degree d surface $S \subset \mathbb{P}^3$.

$\rightarrow$ It is worth working on this bound!

Strategy (2/2)

Let $S : (f = 0) \subset \mathbb{P}^3$ be a smooth irreducible algebraic surface of degree d defined $\mathbb{F}_q$.

Set $C_\Phi^S \stackrel{\text{def}}{=} \{P \in S \mid \Phi(P) \in T_P S\}$. Then $S(\mathbb{F}_q) \subset C_\Phi^S$.

$$C_\Phi^S : f = h = 0 \text{ for } h := X_0^q f_0 + X_1^q f_1 + X_2^q f_2 + X_3^q f_3 \Rightarrow \deg h = d + q - 1.$$

Take a curve $C \subset S$ of degree δ. Then $C(\mathbb{F}_q) \subseteq C \cap C_\Phi^S$.

If $C \cap C_\Phi^S$ is a finite set of points, then

$$\#C(\mathbb{F}_q) \le \frac{\deg(C \cap C_\Phi^S)}{\min_{P \in C(\mathbb{F}_q)} m_P(C, C_\Phi^S)} \le \frac{\delta(d + q - 1)}{2}.$$

Two necessary conditions for $\dim(C \cap C_\Phi^S) = 0$:

Strategy (2/2)

Let $S : (f = 0) \subset \mathbb{P}^3$ be a smooth irreducible algebraic surface of degree d defined $\mathbb{F}_q$.

Set $C_\Phi^S \overset{\text{def}}{=} \{P \in S \mid \Phi(P) \in T_P S\}$. Then $S(\mathbb{F}_q) \subset C_\Phi^S$.

$$C_\Phi^S : f = h = 0 \text{ for } h := X_0^q f_0 + X_1^q f_1 + X_2^q f_2 + X_3^q f_3 \Rightarrow \deg h = d + q - 1.$$

Take a curve $C \subset S$ of degree δ. Then $C(\mathbb{F}_q) \subseteq C \cap C_\Phi^S$.

If $C \cap C_\Phi^S$ is a finite set of points, then

$$\#C(\mathbb{F}_q) \leq \frac{\deg(C \cap C_\Phi^S)}{\min\limits_{P \in C(\mathbb{F}_q)} m_P(C, C_\Phi^S)} \leq \frac{\delta(d+q-1)}{2}.$$

Two necessary conditions for $\dim(C \cap C_\Phi^S) = 0$:

1. $\dim C_\Phi^S = 1$: in this case, the surface is said to be *Frobenius classical*;
   Counterexample: the Hermitian surface $X^{\sqrt{q}+1} + Y^{\sqrt{q}+1} + Z^{\sqrt{q}+1} + T^{\sqrt{q}+1} = 0$ over $\mathbb{F}_q$.

Strategy (2/2)

Let $S : (f = 0) \subset \mathbb{P}^3$ be a smooth irreducible algebraic surface of degree d defined $\mathbb{F}_q$.

Set $C_\Phi^S \stackrel{\text{def}}{=} \{P \in S \mid \Phi(P) \in T_P S\}$. Then $S(\mathbb{F}_q) \subset C_\Phi^S$.

$$C_\Phi^S : f = h = 0 \text{ for } h := X_0^q f_0 + X_1^q f_1 + X_2^q f_2 + X_3^q f_3 \Rightarrow \deg h = d + q - 1.$$

Take a curve $C \subset S$ of degree δ. Then $C(\mathbb{F}_q) \subseteq C \cap C_\Phi^S$.

If $C \cap C_\Phi^S$ is a finite set of points, then

$$\#C(\mathbb{F}_q) \leq \frac{\deg(C \cap C_\Phi^S)}{\min\limits_{P \in C(\mathbb{F}_q)} m_P(C, C_\Phi^S)} \leq \frac{\delta(d + q - 1)}{2}.$$

Two necessary conditions for $\dim(C \cap C_\Phi^S) = 0$:

1. $\dim C_\Phi^S = 1$: in this case, the surface is said to be *Frobenius classical*;

   Counterexample: the Hermitian surface $X^{\sqrt{q}+1} + Y^{\sqrt{q}+1} + Z^{\sqrt{q}+1} + T^{\sqrt{q}+1} = 0$ over $\mathbb{F}_q$.

   ✔ $p \nmid d(d-1) \Rightarrow S$ is Frobenius classical.

Strategy (2/2)

Let $S : (f = 0) \subset \mathbb{P}^3$ be a smooth irreducible algebraic surface of degree d defined $\mathbb{F}_q$.

Set $C_\Phi^S \stackrel{\text{def}}{=} \{P \in S \mid \Phi(P) \in T_P S\}$. Then $S(\mathbb{F}_q) \subset C_\Phi^S$.

$$C_\Phi^S : f = h = 0 \text{ for } h := X_0^q f_0 + X_1^q f_1 + X_2^q f_2 + X_3^q f_3 \Rightarrow \deg h = d + q - 1.$$

Take a curve $C \subset S$ of degree δ. Then $C(\mathbb{F}_q) \subseteq C \cap C_\Phi^S$.

If $C \cap C_\Phi^S$ is a finite set of points, then

$$\#C(\mathbb{F}_q) \leq \frac{\deg(C \cap C_\Phi^S)}{\min\limits_{P \in C(\mathbb{F}_q)} m_P(C, C_\Phi^S)} \leq \frac{\delta(d + q - 1)}{2}.$$

Two necessary conditions for $\dim(C \cap C_\Phi^S) = 0$:

1. $\dim C_\Phi^S = 1$: in this case, the surface is said to be *Frobenius classical*;
   Counterexample: the Hermitian surface $X^{\sqrt{q}+1} + Y^{\sqrt{q}+1} + Z^{\sqrt{q}+1} + T^{\sqrt{q}+1} = 0$ over $\mathbb{F}_q$.
   ✔ $p \nmid d(d-1) \Rightarrow S$ is Frobenius classical.
2. C does not share any components with C_Φ^S.
   Counterexample: if S contains a $\mathbb{F}_q$–line L, then $L \subset C_\Phi^S$. The bound does not hold.

Strategy (2/2)

Let $S : (f = 0) \subset \mathbb{P}^3$ be a smooth irreducible algebraic surface of degree d defined $\mathbb{F}_q$.

Set $C_\Phi^S \overset{\text{def}}{=} \{P \in S \mid \Phi(P) \in T_P S\}$. Then $S(\mathbb{F}_q) \subset C_\Phi^S$.

$$C_\Phi^S : f = h = 0 \text{ for } h := X_0^q f_0 + X_1^q f_1 + X_2^q f_2 + X_3^q f_3 \Rightarrow \deg h = d + q - 1.$$

Take a curve $C \subset S$ of degree δ. Then $C(\mathbb{F}_q) \subseteq C \cap C_\Phi^S$.

If $C \cap C_\Phi^S$ is a finite set of points, then

$$\#C(\mathbb{F}_q) \leq \frac{\deg(C \cap C_\Phi^S)}{\min_{P \in C(\mathbb{F}_q)} m_P(C, C_\Phi^S)} \leq \frac{\delta(d + q - 1)}{2}.$$

Two necessary conditions for $\dim(C \cap C_\Phi^S) = 0$:

1. $\dim C_\Phi^S = 1$: in this case, the surface is said to be *Frobenius classical*;
   Counterexample: the Hermitian surface $X^{\sqrt{q}+1} + Y^{\sqrt{q}+1} + Z^{\sqrt{q}+1} + T^{\sqrt{q}+1} = 0$ over $\mathbb{F}_q$.
   ✔ $p \nmid d(d-1) \Rightarrow S$ is Frobenius classical.
2. C does not share any components with C_Φ^S.
   Counterexample: if S contains a $\mathbb{F}_q$–line L, then $L \subset C_\Phi^S$. The bound does not hold.

Aim: understanding the components of the curve C_Φ^S for a Frobenius classical surface.

Osculating spaces and P-orders (Stöhr–Voloch theory 1)

Let $C \subset \mathbb{P}^3$ be an absolutely irreducible projective curve defined over $\mathbb{F}_q$. Fix $P \in C$.
An integer j is a P–order if there exists a plane intersecting the curve C with multiplicity j at P.
If C is non–plane and P is non–singular, there are exactly four distinct P–orders:

$$j_0 = 0 < j_1 < j_2 < j_3.$$

Remark: $j_1 = 1 \Leftrightarrow C$ is non–singular at the point P.

Osculating spaces and P-orders (Stöhr–Voloch theory 1)

Let $C \subset \mathbb{P}^3$ be an absolutely irreducible projective curve defined over $\mathbb{F}_q$. Fix $P \in C$.
An integer j is a P–order if there exists a plane intersecting the curve C with multiplicity j at P.
If C is non–plane and P is non–singular, there are exactly four distinct P–orders:

$$j_0 = 0 < j_1 < j_2 < j_3.$$

Remark: $j_1 = 1 \Leftrightarrow C$ is non–singular at the point P.

Osculating spaces: $T_P^{(i)}C = \bigcap\{\text{planes } H \text{ s.t. } m_P(C,H) \geq j_{i+1}\}$.

Equation of the osculating plane $T_P^{(2)}C$: $\begin{vmatrix} X_0 & X_1 & X_2 & X_3 \\ x_0 & x_1 & x_2 & x_3 \\ D_t^{(j_1)}x_0 & D_t^{(j_1)}x_1 & D_t^{(j_1)}x_2 & D_t^{(j_1)}x_3 \\ D_t^{(j_2)}x_0 & D_t^{(j_2)}x_1 & D_t^{(j_2)}x_2 & D_t^{(j_2)}x_3 \end{vmatrix} = 0$

where $D_t^{(j)}$ are the *Hasse derivatives* with respect to a a local parameter t at P defined by

$$D_t^{(i)} t^k = \binom{k}{i} t^{k-i}.$$

Frobenius orders (Stöhr–Voloch theory 2)

Fix $P \in C \subset \mathbb{P}^3$ with P–orders $(0, j_1, j_2, j_3)$. Then $\Phi(P) \in T_P^{(2)} C$ if and only if

$$\Delta(j_1, j_2) \overset{\text{def}}{=} \begin{vmatrix} x_0^q & x_1^q & x_2^q & x_3^q \\ x_0 & x_1 & x_2 & x_3 \\ D_t^{(j_1)} x_0 & D_t^{(j_1)} x_1 & D_t^{(j_1)} x_2 & D_t^{(j_1)} x_3 \\ D_t^{(j_2)} x_0 & D_t^{(j_2)} x_1 & D_t^{(j_2)} x_2 & D_t^{(j_2)} x_3 \end{vmatrix} = 0$$

Frobenius orders (Stöhr–Voloch theory 2)

Fix $P \in C \subset \mathbb{P}^3$ with P–orders $(0, j_1, j_2, j_3)$. Then $\Phi(P) \in T_P^{(2)}C$ if and only if

$$\Delta(j_1, j_2) \overset{\text{def}}{=} \begin{vmatrix} x_0^q & x_1^q & x_2^q & x_3^q \\ x_0 & x_1 & x_2 & x_3 \\ D_t^{(j_1)}x_0 & D_t^{(j_1)}x_1 & D_t^{(j_1)}x_2 & D_t^{(j_1)}x_3 \\ D_t^{(j_2)}x_0 & D_t^{(j_2)}x_1 & D_t^{(j_2)}x_2 & D_t^{(j_2)}x_3 \end{vmatrix} = 0$$

Theorem [Stöhr–Voloch, 1986]

There exist integers $\nu_1 < \nu_2$ s.t. $\Delta(\nu_1, \nu_2)$ is a nonzero function.

Definition

The integers $\nu_0 = 0, \nu_1, \nu_2$ chosen minimally with respect to the lexicographic order are called the Frobenius orders of C.

The curve C is Frobenius classical if $(\nu_1, \nu_2) = (1, 2)$, Frobenius non–classical otherwise.

Frobenius non–classical curves on surfaces

Aim: Understand the components of $C_\Phi^S = \{P \in S \mid \Phi(P) \in T_P S\}$ on a Frob. classical surface.

Frobenius non–classical curves on surfaces

Aim: Understand the components of $C_\Phi^S = \{P \in S \mid \Phi(P) \in T_P S\}$ on a Frob. classical surface.

Proposition [BN21]

Let C be a non–plane curve lying on a surface S. Assume that C is Frobenius non–classical with $\nu_1 = 1$. Then C is not a component of C_Φ^S.

Frobenius non–classical curves on surfaces

Aim: Understand the components of $C_\Phi^S = \{P \in S \mid \Phi(P) \in T_P S\}$ on a Frob. classical surface.

Proposition [BN21]

Let C be a non–plane curve lying on a surface S. Assume that C is Frobenius non–classical with $\nu_1 = 1$. Then C is not a component of C_Φ^S.

What about $\nu_1 > 1$?

Frobenius non–classical curves on surfaces

Aim: Understand the components of $C_\Phi^S = \{P \in S \mid \Phi(P) \in T_P S\}$ on a Frob. classical surface.

Proposition [BN21]

Let C be a non–plane curve lying on a surface S. Assume that C is Frobenius non–classical with $\nu_1 = 1$. Then C is not a component of C_Φ^S.

What about $\nu_1 > 1$? $\nu_1 > 1 \Rightarrow \Phi(P) \in T_P C \subset T_P S$

(Sad) Fact: Frobenius non–classical curves with $\nu_1 > 1$ are components of C_Φ^S.

Frobenius non–classical curves on surfaces

Aim: Understand the components of $C_\Phi^S = \{P \in S \mid \Phi(P) \in T_P S\}$ on a Frob. classical surface.

Proposition [BN21]

Let C be a non–plane curve lying on a surface S. Assume that C is Frobenius non–classical with $\nu_1 = 1$. Then C is not a component of C_Φ^S.

What about $\nu_1 > 1$? $\nu_1 > 1 \Rightarrow \Phi(P) \in T_P C \subset T_P S$

(Sad) Fact: Frobenius non–classical curves with $\nu_1 > 1$ are components of C_Φ^S. However...

Proposition [BN21]

Assume that C is Frobenius non–classical with $\nu_1 > 1$ and $\delta \leq q$. Then C is plane.

Remark: Hefez and Voloch (1990) gave the exact number of rational points on **smooth** curves with $\nu_1 > 1$, while Borges and Homma (2018) studied **singular plane** curves with $\nu_1 > 1$.

Frobenius non–classical curves on surfaces

Aim: Understand the components of $C_\Phi^S = \{P \in S \mid \Phi(P) \in T_P S\}$ on a Frob. classical surface.

Proposition [BN21]

Let C be a non–plane curve lying on a surface S. Assume that C is Frobenius non–classical with $\nu_1 = 1$. Then C is not a component of C_Φ^S.

What about $\nu_1 > 1$? $\nu_1 > 1 \Rightarrow \Phi(P) \in T_P C \subset T_P S$

(Sad) Fact: Frobenius non–classical curves with $\nu_1 > 1$ are components of C_Φ^S. However...

Proposition [BN21]

Assume that C is Frobenius non–classical with $\nu_1 > 1$ and $\delta \leq q$. Then C is plane.

Remark: Hefez and Voloch (1990) gave the exact number of rational points on **smooth** curves with $\nu_1 > 1$, while Borges and Homma (2018) studied **singular plane** curves with $\nu_1 > 1$.

Tool: Use the existence and the minimality of the Frobenius orders ν_1, ν_2 s.t. $\Delta(\nu_1, \nu_2) \neq 0$.

Frobenius non–classical curves on surfaces

Aim: Understand the components of $C_\Phi^S = \{P \in S \mid \Phi(P) \in T_P S\}$ on a Frob. classical surface.

Proposition [BN21]

Let C be a non–plane curve lying on a surface S. Assume that C is Frobenius non–classical with $\nu_1 = 1$. Then C is not a component of C_Φ^S.

What about $\nu_1 > 1$? $\nu_1 > 1 \Rightarrow \Phi(P) \in T_P C \subset T_P S$

(Sad) Fact: Frobenius non–classical curves with $\nu_1 > 1$ are components of C_Φ^S. However...

Proposition [BN21]

Assume that C is Frobenius non–classical with $\nu_1 > 1$ and $\delta \leq q$. Then C is plane.

Remark: Hefez and Voloch (1990) gave the exact number of rational points on **smooth** curves with $\nu_1 > 1$, while Borges and Homma (2018) studied **singular plane** curves with $\nu_1 > 1$.

Tool: Use the existence and the minimality of the Frobenius orders ν_1, ν_2 s.t. $\Delta(\nu_1, \nu_2) \neq 0$.
Example: C is Frobenius non–classical $\Rightarrow \{\nu_1, \nu_2\} \neq \{1, 2\} \Rightarrow \Delta(1,2) = 0$. If $\Phi(P) \in T_P S$

$$\Rightarrow \Delta(1,2) = \quad (u'' - g'' u_y) \quad [(x - \tilde{x})g' - (y - \tilde{y})] \quad = 0$$

Frobenius non–classical curves on surfaces

Aim: Understand the components of $C_\Phi^S = \{P \in S \mid \Phi(P) \in T_P S\}$ on a Frob. classical surface.

Proposition [BN21]

Let C be a non–plane curve lying on a surface S. Assume that C is Frobenius non–classical with $\nu_1 = 1$. Then C is not a component of C_Φ^S.

What about $\nu_1 > 1$? $\nu_1 > 1 \Rightarrow \Phi(P) \in T_P C \subset T_P S$

(Sad) Fact: Frobenius non–classical curves with $\nu_1 > 1$ are components of C_Φ^S. However...

Proposition [BN21]

Assume that C is Frobenius non–classical with $\nu_1 > 1$ and $\delta \leq q$. Then C is plane.

Remark: Hefez and Voloch (1990) gave the exact number of rational points on **smooth** curves with $\nu_1 > 1$, while Borges and Homma (2018) studied **singular plane** curves with $\nu_1 > 1$.

Tool: Use the existence and the minimality of the Frobenius orders ν_1, ν_2 s.t. $\Delta(\nu_1, \nu_2) \neq 0$.
Example: C is Frobenius non–classical $\Rightarrow \{\nu_1, \nu_2\} \neq \{1, 2\} \Rightarrow \Delta(1,2) = 0$. If $\Phi(P) \in T_P S$

$$\Rightarrow \Delta(1,2) = \underbrace{(u'' - g'' u_y)}_{\Phi(P) \notin T_P S} \underbrace{[(x - \tilde{x})g' - (y - \tilde{y})]}_{\nu_1 > 1} = 0$$

Frobenius classical components of C_Φ^S

Recap: A component of C_Φ^S falls in one of the following cases:

- $\nu_1 > 1$: in this case, if it has $\delta \leq q$, it is plane;
- it is Frobenius classical, i.e. $\{\nu_1, \nu_2\} = \{1, 2\}$.

Conjecture: Non–plane Frobenius classical curves with $\delta \leq q$ are not components of C_Φ^S.

Frobenius classical components of C_Φ^S

Recap: A component of C_Φ^S falls in one of the following cases:

- $\nu_1 > 1$: in this case, if it has $\delta \leq q$, it is plane;
- it is Frobenius classical, i.e. $\{\nu_1, \nu_2\} = \{1, 2\}$.

Conjecture: Non–plane Frobenius classical curves with $\delta \leq q$ are not components of C_Φ^S.

Example of surface with highly reducible C_Φ^S

Over $\mathbb{F}_5$, consider the surface S defined by

$$\begin{aligned} f = \quad & 2X_0X_1^2 + 2X_1^3 + 2X_0^2X_2 + 2X_0X_1X_2 + X_1^2X_2 + 2X_0X_2^2 + 3X_1X_2^2 \\ & +3X_2^3 + 4X_0^2X_3 + X_0X_1X_3 + X_1^2X_3 + 2X_1X_2X_3 + 2X_2^2X_3 \\ & +3X_0X_3^2 + 4X_1X_3^2 + X_2X_3^2. \end{aligned}$$

The curve C_Φ^S has degree 21 and is formed of 15 $\mathbb{F}_5$–lines and one non–plane sextic $(\delta = q + 1)$.

Main result & Conclusion

Theorem [BN21]

Let S be an irreducible Frobenius classical surface of degree $d > 1$ in $\mathbb{P}^3$. Let C be a non–plane irreducible curve of degree $\delta \leq q$ lying on S. Suppose C is Frobenius non–classical. Then

$$\#C(\mathbb{F}_q) \leq \frac{\delta(d+q-1)}{2}.$$

Under the conjecture, the bound also holds for Frobenius classical curves.

Main result & Conclusion

Theorem [BN21]

Let S be an irreducible Frobenius classical surface of degree $d > 1$ in $\mathbb{P}^3$. Let C be a non–plane irreducible curve of degree $\delta \leq q$ lying on S. Suppose C is Frobenius non–classical. Then

$$\#C(\mathbb{F}_q) \leq \frac{\delta(d+q-1)}{2}.$$

Under the conjecture, the bound also holds for Frobenius classical curves.

- A plane curve on a degree d surface has $\delta \leq d \Rightarrow$ our bound holds for plane curves which have at least one point P such that $\Phi(P) \notin T_P C$ by Stöhr–Voloch bound $(\delta(\delta + q - 1)/2)$.

Main result & Conclusion

Theorem [BN21]

Let S be an irreducible Frobenius classical surface of degree $d > 1$ in $\mathbb{P}^3$. Let C be a non–plane irreducible curve of degree $\delta \leq q$ lying on S. Suppose C is Frobenius non–classical. Then

$$\#C(\mathbb{F}_q) \leq \frac{\delta(d+q-1)}{2}.$$

Under the conjecture, the bound also holds for Frobenius classical curves.

- A plane curve on a degree d surface has $\delta \leq d \Rightarrow$ our bound holds for plane curves which have at least one point P such that $\Phi(P) \notin T_P C$ by Stöhr–Voloch bound $(\delta(\delta+q-1)/2)$.
- **Embedding entails arithmetic and geometric constraints on a variety:**
For $\delta = 11$ and $d = 5$ over $\mathbb{F}_9$, C has genus at most 17 and $\#C(\mathbb{F}_q) \leq 72$.
In `ManyPoints`, maximal curves of genus 16 and 17 have 74 $\mathbb{F}_9$–points.
These record curves cannot lie on a Frobenius classical surface in $\mathbb{P}^3$, unless being a component of C^S_Φ.

Main result & Conclusion

Theorem [BN21]

Let S be an irreducible Frobenius classical surface of degree $d > 1$ in $\mathbb{P}^3$. Let C be a non–plane irreducible curve of degree $\delta \leq q$ lying on S. Suppose C is Frobenius non–classical. Then

$$\#C(\mathbb{F}_q) \leq \frac{\delta(d+q-1)}{2}.$$

Under the conjecture, the bound also holds for Frobenius classical curves.

Future questions

Main result & Conclusion

Theorem [BN21]

Let S be an irreducible Frobenius classical surface of degree $d > 1$ in $\mathbb{P}^3$. Let C be a non–plane irreducible curve of degree $\delta \leq q$ lying on S. Suppose C is Frobenius non–classical. Then

$$\#C(\mathbb{F}_q) \leq \frac{\delta(d+q-1)}{2}.$$

Under the conjecture, the bound also holds for Frobenius classical curves.

Future questions

- Prove the conjecture!

Main result & Conclusion

Theorem [BN21]

Let S be an irreducible Frobenius classical surface of degree $d > 1$ in $\mathbb{P}^3$. Let C be a non–plane irreducible curve of degree $\delta \leq q$ lying on S. Suppose C is Frobenius non–classical. Then

$$\#C(\mathbb{F}_q) \leq \frac{\delta(d+q-1)}{2}.$$

Under the conjecture, the bound also holds for Frobenius classical curves.

Future questions

- Prove the conjecture!
- Our theorem essentially relies on the geometry of space curves and the intersection theory in $\mathbb{P}^3$.

  Can we generalize our approach when $C \subset S \subset \mathbb{P}^n$, for $n \geq 4$?

Main result & Conclusion

Theorem [BN21]

Let S be an irreducible Frobenius classical surface of degree $d > 1$ in $\mathbb{P}^3$. Let C be a non–plane irreducible curve of degree $\delta \leq q$ lying on S. Suppose C is Frobenius non–classical. Then

$$\#C(\mathbb{F}_q) \leq \frac{\delta(d+q-1)}{2}.$$

Under the conjecture, the bound also holds for Frobenius classical curves.

Future questions and job alert

- Prove the conjecture!
- Our theorem essentially relies on the geometry of space curves and the intersection theory in $\mathbb{P}^3$.

  Can we generalize our approach when $C \subset S \subset \mathbb{P}^n$, for $n \geq 4$?

Main result & Conclusion

Theorem [BN21]

Let S be an irreducible Frobenius classical surface of degree $d > 1$ in $\mathbb{P}^3$. Let C be a non–plane irreducible curve of degree $\delta \leq q$ lying on S. Suppose C is Frobenius non–classical. Then

$$\#C(\mathbb{F}_q) \leq \frac{\delta(d+q-1)}{2}.$$

Under the conjecture, the bound also holds for Frobenius classical curves.

Future questions and job alert

- Prove the conjecture!
- Our theorem essentially relies on the geometry of space curves and the intersection theory in $\mathbb{P}^3$.

  Can we generalize our approach when $C \subset S \subset \mathbb{P}^n$, for $n \geq 4$?

Thank you for your attention!

What about $C \subset S \subset \mathbb{P}^n$ for $n \geq 4$?

Can we generalize our approach when $C \subset S \subset \mathbb{P}^n$, for $n \geq 4$?

What about $C \subset S \subset \mathbb{P}^n$ for $n \geq 4$?

Can we generalize our approach when $C \subset S \subset \mathbb{P}^n$, for $n \geq 4$?

Consider the varieties in $S \times \mathbb{P}^n$

- $\Gamma_C = \{(P, \Phi(P)) \in C^2 \mid P \in C\}$ the graph of Φ restricted to the curve C,
- $\mathcal{T}_S = \{(P, Q) \in S \times \mathbb{P}^n \mid P \in S,\ Q \in T_P S\}$.

Then $C(\mathbb{F}_q) \xhookrightarrow{\Delta} \Gamma_C \cap \mathcal{T}_S \simeq \{P \in C \mid \Phi(P) \in T_P S\}$.
Remark: C_Φ^S was the image of $\Gamma_C \cap \mathcal{T}_S \in S \times \mathbb{P}^3$ under the 1^{st} projection.

What about $C \subset S \subset \mathbb{P}^n$ for $n \geq 4$?

Can we generalize our approach when $C \subset S \subset \mathbb{P}^n$, for $n \geq 4$?

Consider the varieties in $S \times \mathbb{P}^n$

- $\Gamma_C = \{(P, \Phi(P)) \in C^2 \mid P \in C\}$ the graph of Φ restricted to the curve C, (dim 1)
- $\mathcal{T}_S = \{(P, Q) \in S \times \mathbb{P}^n \mid P \in S,\ Q \in T_P S\}$. (dim 4)

Then $C(\mathbb{F}_q) \xhookrightarrow{\Delta} \Gamma_C \cap \mathcal{T}_S \simeq \{P \in C \mid \Phi(P) \in T_P S\}$.
Remark: C^S_Φ was the image of $\Gamma_C \cap \mathcal{T}_S \in S \times \mathbb{P}^3$ under the 1^{st} projection.

Γ_C and $\mathcal{T}_S$ have complementary dimensions in $S \times \mathbb{P}^n$ (of dim $n+2$) *if and only if* $n = 3$.
$\rightarrow$ bound the number of rational points on C by a fraction of the intersection product $[\Gamma_C] \cdot [\mathcal{T}_S]$.

What about $C \subset S \subset \mathbb{P}^n$ for $n \geq 4$?

Can we generalize our approach when $C \subset S \subset \mathbb{P}^n$, for $n \geq 4$?

Consider the varieties in $S \times \mathbb{P}^n$

- $\Gamma_C = \{(P, \Phi(P)) \in C^2 \mid P \in C\}$ the graph of Φ restricted to the curve C, (dim 1)
- $\mathcal{T}_S = \{(P, Q) \in S \times \mathbb{P}^n \mid P \in S,\ Q \in T_P S\}$. (dim 4)

Then $C(\mathbb{F}_q) \xhookrightarrow{\Delta} \Gamma_C \cap \mathcal{T}_S \simeq \{P \in C \mid \Phi(P) \in T_P S\}$.
Remark: C_Φ^S was the image of $\Gamma_C \cap \mathcal{T}_S \in S \times \mathbb{P}^3$ under the 1^{st} projection.

Γ_C and $\mathcal{T}_S$ have complementary dimensions in $S \times \mathbb{P}^n$ (of dim $n+2$) *if and only if* $n = 3$.
$\rightarrow$ bound the number of rational points on C by a fraction of the intersection product $[\Gamma_C] \cdot [\mathcal{T}_S]$.

When $n \geq 4$, $[\Gamma_C] \cdot [\mathcal{T}_S] = 0$ while $\Gamma_C \cap \mathcal{T}_S \neq \varnothing$.

What about $C \subset S \subset \mathbb{P}^n$ for $n \geq 4$?

Can we generalize our approach when $C \subset S \subset \mathbb{P}^n$, for $n \geq 4$?

Consider the varieties in $S \times \mathbb{P}^n$

- $\Gamma_C = \{(P, \Phi(P)) \in C^2 \mid P \in C\}$ the graph of Φ restricted to the curve C, (dim 1)
- $\mathcal{T}_S = \{(P, Q) \in S \times \mathbb{P}^n \mid P \in S,\ Q \in T_P S\}$. (dim 4)

Then $C(\mathbb{F}_q) \xhookrightarrow{\Delta} \Gamma_C \cap \mathcal{T}_S \simeq \{P \in C \mid \Phi(P) \in T_P S\}$.
Remark: C_Φ^S was the image of $\Gamma_C \cap \mathcal{T}_S \in S \times \mathbb{P}^3$ under the 1^{st} projection.

Γ_C and $\mathcal{T}_S$ have complementary dimensions in $S \times \mathbb{P}^n$ (of dim $n+2$) *if and only if* $n = 3$.
$\rightarrow$ bound the number of rational points on C by a fraction of the intersection product $[\Gamma_C] \cdot [\mathcal{T}_S]$.

When $n \geq 4$, $[\Gamma_C] \cdot [\mathcal{T}_S] = 0$ while $\Gamma_C \cap \mathcal{T}_S \neq \varnothing$.

Idea: Fix this dimension incompatibility by blowing up $\mathcal{T}_S$ or $S \times S$.