New MRD codes from linear cutting blocking sets

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joint works with G. Marino, A. Neri, L. Vicino

Outline

1. Rank metric codes

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2. Generalized rank weights

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2. Generalized rank weights
3. Minimal Rank metric codes via “scattered-ish” objects

Rank-metric codes

$$V = V(n, q^m) = \mathbb{F}_{q^m}^n = \mathbb{F}_{q^m} \times \cdots \times \mathbb{F}_{q^m}$$

$$v = (v_1, \ldots, v_n) \in V, \qquad u = (u_1, \ldots, u_n) \in V$$

$$\begin{aligned} \mathrm{wt}_{\mathrm{rk}}(v) &= \dim_{\mathbb{F}_q} \langle v_1, \ldots, v_n \rangle_{\mathbb{F}_q} && \textbf{rank weight} \text{ of } v \\ \mathrm{d}_{\mathrm{rk}}(u, v) &= \mathrm{wt}_{\mathrm{rk}}(u - v) && \textbf{rank distance} \end{aligned}$$

Rank-metric codes

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Definition (Delsarte 1978 - Gabidulin 1985)

An $[n, k]_{q^m/q}$ **(rank-metric) code** is a k-dimensional $\mathbb{F}_{q^m}$-subspace of $V(n, q^m)$ endowed with the rank distance.

Generalized rank weights

$\Gamma := \{\gamma_1, \ldots, \gamma_m\}$ $\mathbb{F}_q$-*basis of* $\mathbb{F}_{q^m}$
$\Gamma(v) \in \mathbb{F}_q^{m \times n}$ *whose columns are the coordinates of* v_i *wrt* Γ

$$\mathrm{wt}_{\mathsf{rk}}(v) = \mathsf{rk}(\Gamma(v))$$

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$$\mathrm{wt}_{\mathrm{rk}}(v) = \mathrm{rk}(\Gamma(v))$$

$$\sigma_\Gamma(v) = \langle \text{rows of } \Gamma(v) \rangle_{\mathbb{F}_q} \quad \Gamma\text{-}\textbf{support} \text{ of } v$$

The Γ-support does not depend on the choice of Γ

Definition

$\sigma^{rk}(v) := \sigma_\Gamma(v)$ **(rank) support** of v

$$\mathrm{wt}_{\mathrm{rk}}(v) = \dim_{\mathbb{F}_q} \sigma^{rk}(v)$$

Kurihara, Uyematsu, Matsumoto 2012, 2015
Oggier, Sboui 2012 - Jurrius, Pellikaan 2017

Generalized rank weights

Definition

Let $\mathcal{D}$ be an $\mathbb{F}_{q^m}$-subcode of an $[n,k]_{q^m/q}$ code $\mathcal{C}$.

$\sigma^{\mathrm{rk}}(\mathcal{D})$	**rank support** of $\mathcal{D}$	$\mathbb{F}_q$-subspace spanned by the rank support of all its codewords
$\mathrm{wt}_{\mathrm{rk}}(\mathcal{D})$	**rank weight** of $\mathcal{D}$	the dimension of $\sigma^{\mathrm{rk}}(\mathcal{D})$

Definition

$[n,k]_{q^m/q}$ code $\mathcal{C}$ $\qquad 1 \leq j \leq k$

$$d_{\mathrm{rk},j} = \min\{\mathrm{wt}_{\mathrm{rk}}(\mathcal{D}) : \ \mathcal{D} \subseteq \mathcal{C}, \dim \mathcal{D} = j\}$$

j-th generalized rank weight of $\mathcal{C}$

Rank-metric codes

$[n, k, (d_1, \ldots, d_k)]_{q^m/q}$ code $\mathcal{C}$ $\qquad$ $d_i := d_{\mathrm{rk},i}(\mathcal{C})$.

$$d(\mathcal{C}) := d_1 = \min\{\mathrm{wt}_{\mathrm{rk}}(v) : \ v \in \mathcal{C}, v \neq 0\}$$

$\mathcal{C}$ *is an* $[n, k, d]_{q^m/q}$ *code*

Theorem

1. $mk \leq \min\{m(n - d_1 + 1), n(m - d_1 + 1)\}$

   Singleton Bound - Delsarte 1978

2. $d_s \leq \min\left\{n - k + s, sm, \frac{m}{n}(n - k) + m(s - 1) + 1\right\}$

   U. Martínez-Peñas 2016

Definition

$$mk = \min\{m(n - d_1 + 1), n(m - d_1 + 1)\}$$
$$\Downarrow$$
$\mathcal{C}$ maximum rank distance (MRD)

q-systems

Definition

$U \leq_{\mathbb{F}_q} V(k, q^m)$ $\qquad$ $H \leq_{\mathbb{F}_{q^m}} V(k, q^m)$

$$\mathrm{wt}_U(H) := \dim_{\mathbb{F}_q}(H \cap U)$$

weight of H in U

Definition

$$\begin{array}{l} U \leq_{\mathbb{F}_q} V(k, q^m) \\ \dim_{\mathbb{F}_q}(U) = n \\ \langle U \rangle_{\mathbb{F}_{q^m}} = V(k, q^m) \end{array} \Rightarrow \begin{array}{l} [n, k, (d_1, \ldots, d_k)]_{q^m/q} \text{ system if} \\ \forall \quad i \in \{1, \ldots, k\} \\ d_i := n - \max\left\{ \mathrm{wt}_U(H) : \begin{array}{c} H \subseteq V(k, q^m) \\ \dim_{\mathbb{F}_{q^m}}(H) = k - i \end{array} \right\} \end{array}$$

$[n, k, (d_1, \ldots, d_k)]_{q^m/q}$ **system** $U \iff [n, k, (d_1, \ldots, d_k)]_{q^m/q}$ **code** $\mathcal{C}_U$

Evasive subspace

Definition (B., Csajbók, Marino, Trombetti 2021)

$h < k$

$[n,k]_{q^m/q}$ system U
$(h,r)_q$-**evasive subspace**
(or simply $(h,r)_q$-**evasive**)

$\iff$

$\langle U \rangle_{\mathbb{F}_{q^m}} = V(k,q^m)$,
$\dim_{\mathbb{F}_q}(U \cap H) \leq r$
$\forall$ $\mathbb{F}_{q^m}$-subspace $H \subset V(k,q^m)$
with $\dim_{\mathbb{F}_{q^m}}(H) = h$

$(h,h)_q$-evasive subspace $\longrightarrow$ h-**scattered**
$(1,1)_q$-evasive subspace $\longrightarrow$ **scattered**

Evasive subspace and scattered subspaces

Theorem (Blokhuis, Lavrauw 2000 - Csajbók, Marino, Polverino, Zullo 2021)

$$U \subset V(k, q^m) \text{ } h\text{-scattered} \Longrightarrow \dim_{\mathbb{F}_q}(U) \leq \frac{km}{h+1}$$

When the previous equality is reached, U is said be **maximum**.

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Theorem (Zini, Zullo 2021)

$n := \frac{km}{h+1}$ $\qquad$ $[n, k]_{q^m/q}$ *system* U

$$U \text{ maximum } h\text{-scattered} \iff \mathcal{C}_U \text{ MRD}$$

Linear cutting q-systems

Definition

$[n,k]_{q^m/q}$ code $\mathcal{C}$

$$v \in \mathcal{C} \textbf{ minimal codeword} \iff \begin{array}{l} \forall\ v' \in \mathcal{C},\ \sigma^{\mathrm{rk}}(v') \subseteq \sigma^{\mathrm{rk}}(v) \\ \text{implies } v' = \alpha v \text{ for some } \alpha \in \mathbb{F}_{q^m} \end{array}$$

$\mathcal{C}$ is **minimal** if all its codewords are minimal

Definition

$$\begin{array}{c} [n,k]_{q^m/q} \text{ system } U \\ t\textbf{-cutting} \end{array} \iff \begin{array}{c} \forall\ \mathbb{F}_{q^m}\text{-subspace } H \subset V(k,q^m) \\ \text{of codimension } t \\ \langle H \cap U \rangle_{\mathbb{F}_{q^m}} = H \end{array}$$

When $t = 1$, we simply say that U is **(linear) cutting**

Theorem

$$[n,k]_{q^m/q} \textit{ system } U \textit{ cutting} \iff \mathcal{C}_U \textit{ minimal rank-metric code}$$

Alfarano, Borello, Neri, Ravagnani 2022

Evasive subspaces and cutting q-systems

- $[n,k]_{q^m/q}$ system U cutting, $k \geq 2$ $\implies n \geq m+k-1$
- $[n,3]_{q^m/q}$ system U scattered, $n \geq m+2$ $\implies U$ cutting

Alfarano, Borello, Neri, Ravagnani 2022

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Alfarano, Borello, Neri, Ravagnani 2022

- $[n,k]_{q^m/q}$ system U

$$(k-2, n-m-1)_q\text{-evasive} \iff \text{cutting}$$

- nondegenerate $[n,k,(d_1,\dots,d_k)]_{q^m/q}$ code $\mathcal{C}$

$$\mathcal{C} \text{ minimal} \iff d_2 \geq m+1$$

- $m < (k-1)^2 \implies$ no cutting $[m+k-1,k]_{q^m/q}$ systems

Cutting q-systems and rank-metric codes

$$\begin{array}{c}[n,k]_{q^m/q} \text{ system } U \\ \text{cutting, } k \geq 2\end{array} \implies n \geq m + k - 1$$

Problem

Construct minimal rank-metric codes

- *reaching the lower bound on the dimension of the associated q-system*
- *with the maximum value for the corresponding values of generalized rank weights*

Cutting $[n,3]_{q^m/q}$ systems

Minimal $[n,3]_{q^m/q}$ codes $\Longleftrightarrow$ cutting $[n,3]_{q^m/q}$ systems

$\mathbb{F}_q$-scattered subspace in $V(3,q^m)$ of dimension $n \geq m+2$

Cutting $[n,3]_{q^m/q}$ systems

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$\mathbb{F}_q$-scattered subspace in $V(3,q^m)$ of dimension $n \geq m+2$

- (Ball, B., Blokhuis, Csajbók, Giulietti, Lavrauw, Marino, Polverino, Zullo)
  $m \geq 4$ even
  there always exists maximum scattered in $V(3,q^m)$
  (and hence scattered of dimension $m+2$)
- (Blokhuis, Lavrauw 2000)
  $m \not\equiv 3,5 \pmod 6$ and $m \geq 4$
  there are scattered in $V(3,q^m)$ of dimension $m+2$
- (B., Csajbók, Marino, Trombetti 2021)
  There are examples of scattered in $V(3,q^5)$ of dimension 7

Cutting $[n,4]_{q^m/q}$ systems, $m = 3, 4$

$c_q(k,m)$:= smallest dimension of a linear cutting $[n,k]_{q^m/q}$ system

- $c_q(k,m) \geq k + m - 1$
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$(m = 3)$

$$c_q(4,3) \geq 7$$

- *There are no cutting* $[7,4]_{q^3/q}$ *systems*
- $U = \{(\alpha_0 + \alpha_1 u, \alpha_2 + \alpha_3 u, \alpha_4 + \alpha_5 u, \alpha_6 + \alpha_7 u) \ : \ \alpha_i \in \mathbb{F}_q\}$ *is a cutting* $[8,4]_{q^3/q}$ *system.*

$$\Downarrow$$

$$c_q(4,3) = 8$$

Cutting $[n, 4]_{q^4/q}$ systems

$$c_q(4, 4) \geq 8$$

Cutting $[n,4]_{q^4/q}$ systems

$$c_q(4,4) \geq 8$$

$$U := \left\{ \left(x, y, x^q + y^{q^2}, x^{q^2} + y^q + y^{q^2} \right) \, : \, x, y \in \mathbb{F}_{q^4} \right\} \subset \mathbb{F}_{q^4}^4$$

Theorem

$$\begin{array}{c} U \leq_{\mathbb{F}_q} \mathbb{F}_{q^4}^4 \\ \textit{maximum scattered} \end{array} \iff \begin{array}{c} q = 2^h \\ h \not\equiv 2 \pmod 4 \end{array}$$

$$U \implies [8,4,3]_{q^4/q} \text{ MRD code } \mathcal{C}_U$$

Case $(k, m) = (4, 4)$

direct sum of two $[4, 2, 3]_{q^4/q}$ *MRD codes* $\Longrightarrow$ $[8, 4, 3]_{q^4/q}$ *MRD codes*

Theorem

$$\begin{array}{c} q = 2^h \\ h \equiv 1 \pmod 2 \end{array} \Longrightarrow U \text{ is } (2,3)\text{-evasive}$$

$$\Downarrow$$

$\mathcal{C}_U$ *minimal* $[8, 4, 3]_{q^4/q}$ *MRD code*

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Code	is MRD?	$d_{rk,1}$	$d_{rk,2}$	$d_{rk,3}$	$d_{rk,4}$	is minimal?
$\mathcal{C}$	yes	3	5	7	8	yes
$\mathcal{D}_1 \oplus \mathcal{D}_2$	yes	3	4	7	8	no

Indecomposable scattered sequences

Definition (B. Marino,Neri, Vicino 2022)

$i_1, i_2, \ldots, i_m \in \mathbb{N}$
$f_1, \ldots, f_s$ linearized q-polynomials over $\mathbb{F}_{q^n}$ in $\underline{X} = (X_1, \ldots, X_m)$

$\mathcal{F} := (f_1, \ldots, f_s)$ is $(\mathcal{I}; h)_{q^n}$-**scattered sequence** of order m

$$\Updownarrow$$

$$U_{\mathcal{F}} := \left\{ \begin{array}{c} (x_1^{q^{i_1}}, \ldots, x_m^{q^{i_m}}, f_1(\underline{x}), \ldots, f_s(\underline{x})) : \\ \underline{x} = (x_1, \ldots, x_m) \in \mathbb{F}_{q^n}^m \end{array} \right\} \subseteq_{\mathbb{F}_q} V(s+m, q^n)$$

is maximum h-scattered in $V(m+s, q^n)$,

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is maximum h-scattered in $V(m+s, q^n)$,

$f(X)$ scattered polynomial of index t $\iff$ $\{(x^{q^t}, f(x)) : x \in \mathbb{F}_{q^n}\} \subseteq_{\mathbb{F}_q} V(2, q^n)$ scattered

Sheekey, B.-Zhou

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is maximum h-scattered in $V(m+s, q^n)$,

Definition

$(\mathcal{I}; h)_{q^n}$-**scattered sequence** $\mathcal{F} := (f_1, \ldots, f_s)$ of order m is **indecomposable** if it not the direct sum of two smaller scattered sequences

Indecomposable scattered sequences

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$f_1, \ldots, f_s$ linearized q-polynomials over $\mathbb{F}_{q^n}$ in $\underline{X} = (X_1, \ldots, X_m)$

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is maximum h-scattered in $V(m+s, q^n)$,

Definition

$(\mathcal{I}; h)_{q^n}$-**scattered sequence** $\mathcal{F} := (f_1, \ldots, f_s)$ of order m is **exceptional** if it is h-scattered over infinitely many extensions $\mathbb{F}_{q^{n\ell}}$ of $\mathbb{F}_{q^n}$

Indecomposable scattered sequences

$$U := \left\{ \left(x, y, x^q + y^{q^2}, x^{q^2} + y^q + y^{q^2} \right) \, : \, x, y \in \mathbb{F}_{q^4} \right\} \subset \mathbb{F}_{q^4}^4$$

indecomposable $((0,0),1)_{q^4}$-scattered sequence
of order larger than one for $q = 2^{2s+1}$

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Definition

$\alpha, \beta, \gamma \in \mathbb{F}_{q^n}^*$, and $I \neq J \in \mathbb{N}$, $I, J < n-1$

$$U_{\alpha,\beta,\gamma}^{I,J,n} := \left\{ \left(x, y, x^{q^I} + \alpha y^{q^J}, x^{q^J} + \beta y^{q^I} + \gamma y^{q^J} \right) \, : \, x, y \in \mathbb{F}_{q^n} \right\}.$$

Indecomposable scattered sequences

Theorem

$\gcd(I, J, n) = 1$

$$P^{I,J}_{\alpha,\beta,\gamma}(X) := \begin{cases} X^{q^{J-I}+1} + \gamma X - \alpha\beta, & \text{if } I < J, \\ X^{q^{I-J}+1} + \gamma X^{q^{I-J}} - \alpha\beta, & \text{if } I > J, \end{cases} \tag{1}$$

$P^{I,J}_{\alpha,\beta,\gamma}(X)$ *no roots in* $\mathbb{F}_{q^n} \implies$ $U^{I,J,n}_{\alpha,\beta,\gamma}$ *exceptional scattered*, $U^{I,J,n}_{\alpha,\beta,\gamma}$ $(2, 2\max\{I, J\})_q$*-evasive*

Corollary

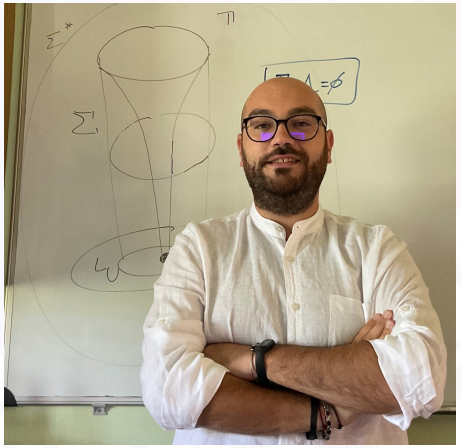
$P^{I,J}_{\alpha,\beta,\gamma}(X)$ *no roots in* $\mathbb{F}_{q^n}$, $\max\{I, J\} \leq (n-1)/2$ $\implies U^{I,J,n}_{\alpha,\beta,\gamma}$ *indecomposable cutting exceptional scattered*

Previous connections with algebraic varieties

- Exceptional Scattered polynomials

B.-Zhou, Ferraguti-Micheli, B.-Montanucci, B.-Giulietti-Zini

- Exceptional MRD codes



B. - Zini- Zullo

Link with algebraic varieties

- $f(x)$ is scattered $\iff$ $rank \begin{pmatrix} x & f(x) \\ y & f(y) \end{pmatrix} = 2$ unless $x = \lambda y$ with $\lambda \in \mathbb{F}_q$

$$\frac{Xf(Y) - Yf(X)}{X^q Y - XY^q} = 0 \rightarrow \text{plane curve}$$

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$$\frac{Xf(Y)-Yf(X)}{X^qY-XY^q} = 0 \to \text{plane curve}$$

- $(x^{q^l} + \alpha y^{q^J}, x^{q^J} + \beta y^{q^l} + \gamma y^{q^J})$ is $((0,0),1)_{q^4}$-scattered sequence

$$\iff rank \begin{pmatrix} x & y & x^{q^l} + \alpha y^{q^J} & x^{q^J} + \beta y^{q^l} + \gamma y^{q^J} \\ z & t & z^{q^l} + \alpha t^{q^J} & z^{q^J} + \beta t^{q^l} + \gamma t^{q^J} \end{pmatrix} = 2$$

unless $(x,y) = \lambda(z,t)$ with $\lambda \in \mathbb{F}_q$

$$\begin{cases} xt - zy = 0 \\ x(z^{q^l} + \alpha t^{q^J}) - z(x^{q^l} + \alpha y^{q^J}) = 0 \\ x(z^{q^J} + \beta t^{q^l} + \gamma t^{q^J}) - z(x^{q^J} + \beta y^{q^l} + \gamma y^{q^J}) = 0 \end{cases} \to \text{space curve}$$

Open problems

Open Problem 1. Find other example of scattered sequences of order > 2

Open Problem 2. Investigate the space curve arising from (f, g) to obtain nonexistence results

Open Problem 3. Investigate the generalized weights of these new objects

THANK YOU

FOR YOUR ATTENTION