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Complete verification of strong BSD for some absolutely simple RM abelian surfaces

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joint with Michael Stoll

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The BSD conjecture

Fundamental problems

Let A be an abelian variety over $\mathbf{Q}$.

Problem 1
Compute $r := \operatorname{rk} A(\mathbf{Q})$, the *algebraic rank*.

For every $n > 1$, there is an *n*-descent exact sequence

$$0 \to A(\mathbf{Q})/n \to \mathrm{Sel}_n(A/\mathbf{Q}) \to Ш(A/\mathbf{Q})[n] \to 0$$

with the n-Selmer group $\mathrm{Sel}_n(A/\mathbf{Q})$ finite
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Problem 2
Compute $Ш(A/\mathbf{Q})$, the *Tate–Shafarevich group*.

Statement of the BSD conjecture

Birch–Swinnerton-Dyer (rank) conjecture

$$r = r_{\mathrm{an}} := \mathrm{ord}_{s=1} L(A, s)$$

For $A = E$ an elliptic curve:

- r_{an} well-defined by modularity of $E/\mathbf{Q}$.
- Yields "day-night algorithm" to compute r and hence $E(\mathbf{Q})$.
- Formulated based on computations in 1965.
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strong BSD conjecture

$$\#\text{Ш}(A/\mathbf{Q}) = \#\text{Ш}(A/\mathbf{Q})_{\mathrm{an}} := \frac{\#A(\mathbf{Q})_{\mathrm{tors}} \cdot \#A^{\vee}(\mathbf{Q})_{\mathrm{tors}}}{\prod_p c_p} \cdot \frac{L^*(A, 1)}{\Omega_A \operatorname{Reg}_A}$$

Compare with the analytic class number formula!

What is known?

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Let A be an RM abelian variety over $\mathbf{Q}$ with associated newform f.

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- This implies by combining the Gross–Zagier formula with the Heegner point Euler system of Kolyvagin–Logachëv:

$$\begin{aligned} r &= r_{\mathrm{an}}, \quad \text{(BSD rank conjecture)} \\ \#Ш(A/\mathbf{Q}) &< \infty, \\ \#Ш(A/\mathbf{Q})_{\mathrm{an}} &\in \mathbf{Q}_{>0}. \end{aligned}$$

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$$\#\text{Ш}(A/\mathbf{Q}) < \infty,$$
$$\#\text{Ш}(A/\mathbf{Q})_{\mathrm{an}} \in \mathbf{Q}_{>0}.$$

- Unknown: $\#\text{Ш}(A/\mathbf{Q}) \stackrel{?}{=} \#\text{Ш}(A/\mathbf{Q})_{\mathrm{an}}$ (strong BSD)

In which cases has strong BSD been verified?

- For elliptic curves with $r_{\mathrm{an}} \leq 1$:
  - Strong BSD verified exactly for levels $N < 5000$ combining work of Grigorov–Jorza–Patrikis–Stein–Tarniţă (2009), Miller (2011), Miller–Stoll (2013, isogeny descent), Creutz–Miller (2012, second isogeny descent), Lawson–Wuthrich (2016, use of p-adic L-functions).

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- For RM abelian varieties of dimension > 1:
  - Flynn–Leprévost–Schaefer–Stein–Stoll–Wetherell (2001): BSD for some Jacobians of dimension 2 numerically.
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  - Skinner–Urban/Skinner (2014/16):
    GL_2 Iwasawa Main Conjecture (IMC) for primes p of good ordinary and bad multiplicative reduction and ρ_p irreducible.

Our results in dimension 2

How to bound $\#Ш(A/\mathbf{Q})$?

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Solution to the "horizontal" problem:

Theorem: explicit Euler system of Kolyvagin–Logachëv
Let A be an RM abelian variety over $\mathbf{Q}$.

One has $\text{Ш}(A/\mathbf{Q})[\mathfrak{p}] = 0$ for all $\mathfrak{p}$ with

- $\rho_\mathfrak{p} : \mathrm{Gal}(\overline{\mathbf{Q}}|\mathbf{Q}) \to \mathrm{Aut}_{\mathbf{F}_\mathfrak{p}}(A[\mathfrak{p}](\overline{\mathbf{Q}}))$ irreducible and
- $\mathfrak{p} \nmid 2 \cdot c \cdot \mathrm{gcd}_K(I_K)$ with Heegner indices I_K and the Tamagawa product c.

What are the main obstacles in dimension > 1?

Problems when $\dim A > 1$ (necessary input for Euler system)

- $\rho_{\mathfrak{p}}$: We don't have an analog of Mazur's classification of rational isogenies of prime degree for *all* A: moduli spaces have dimension > 1.
- I_K: We have to compute Heegner points.

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We solve the problems for concretely given $A = \mathrm{Jac}(C)$.

How to compute the remaining $\text{Ш}(A/\mathbf{Q})[\mathfrak{p}^\infty]$?

Two tools:

- Perform a $\mathfrak{p}^n$-descent to compute $\mathrm{Sel}_{\mathfrak{p}^n}(A/\mathbf{Q})$.
  - Works very well if $\rho_{\mathfrak{p}^n}$ is reducible.
  - Works for general $\mathfrak{p}^n$ in principle, but:
  - Infeasible if $\rho_{\mathfrak{p}^n}$ has large image,
    e.g., $\#\mathcal{O}/\mathfrak{p}^n > 7$ and $\rho_{\mathfrak{p}^n}$ irreducible, even assuming GRH.

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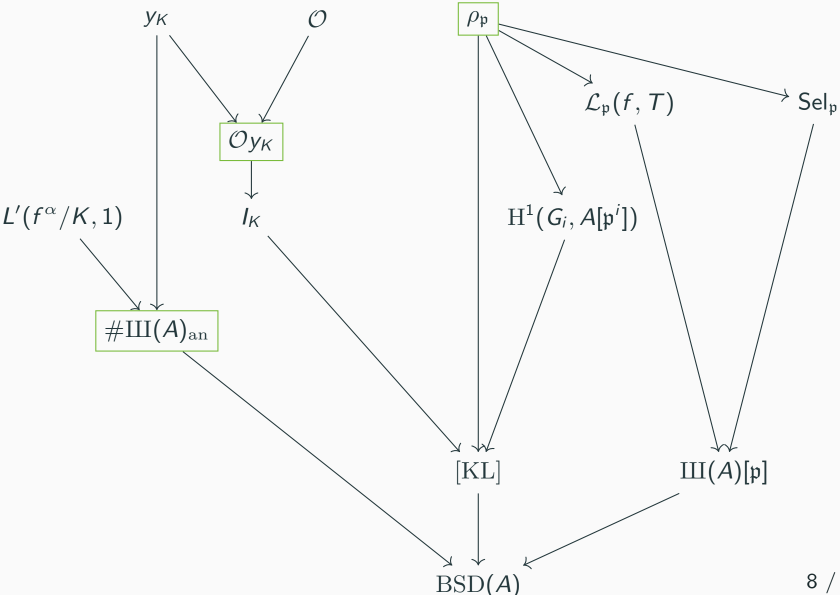
    e.g., $\#\mathcal{O}/\mathfrak{p}^n > 7$ and $\rho_{\mathfrak{p}^n}$ irreducible, even assuming GRH.
- Compute the $\mathfrak{p}$-adic L-function and use the GL_2 IMC.
  - Can be computed very efficiently with overconvergent modular symbols using the Pollack–Stevens–Greenberg algorithm.
  - Requires $\rho_{\mathfrak{p}}$ to be irreducible.
    (But: work in progress joint with Yin)
  - Requires the computation of the $\mathfrak{p}$-adic regulator if $r_{\mathrm{an}} > 0$ or if the reduction is split multiplicative.
    (work in progress by Kaya–Masdeu–Müller–van der Put)

How do we verify the conjecture?



Sketch of some algorithms

Almost all $\rho_{\mathfrak{p}}$ are irreducible

Theorem
Assume $v_p(N) \leq 1$.
If $\rho_{\mathfrak{p}}$ is reducible, $\rho_{\mathfrak{p}}^{\mathrm{ss}} \cong \varepsilon \oplus \varepsilon^{-1}\chi_p$ with $\varepsilon\colon (\mathbf{Z}/d)^\times \to \mathbf{F}_{\mathfrak{p}}^\times$ of conductor d with $d^2 \mid N$.

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an eigenvalue of $\rho_{\mathfrak{p}}(\mathrm{Frob}_\ell)$ has order dividing $\operatorname{ord}(\bar{\ell} \in (\mathbf{Z}/d)^\times)$.

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Hence:

$$\mathrm{res}_{\mathbf{F}_\mathfrak{p}[X]}\left(\mathrm{charpol}_{\mathbf{F}_\mathfrak{p}[X]}(\rho_\mathfrak{p}(\mathrm{Frob}_\ell)), X^{\mathrm{ord}(\bar{\ell} \in (\mathbf{Z}/d)^\times)} - 1\right) = 0.$$

for d maximal with $d^2 \mid N$.

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Hence:

$$\mathfrak{p} \mid \mathrm{res}_{\mathcal{O}[X]}\left(\mathrm{charpol}_{\mathcal{O}[X]}(\rho_{\mathfrak{p}^\infty}(\mathrm{Frob}_\ell)), X^{\mathrm{ord}(\bar{\ell}\in(\mathbf{Z}/d)^\times)} - 1\right).$$

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for d maximal with $d^2 \mid N$.

Computing (a multiple of) the Heegner index I_K

Let $J = \mathrm{Jac}(X)$. There is an isogeny
$\pi\colon J_0(N)/\mathrm{Ann}_{\mathbf{T}}(f) =: A_f \to J$.

Let K be a Heegner field for J.

$$\begin{array}{ccccc} A_f(K) & \hookrightarrow & A_f(\mathbf{C}) & \xrightarrow{\sim} & \mathbf{C}^g/\Lambda_f \\ \downarrow & & \downarrow & & \downarrow \pi \\ J(K) & \hookrightarrow & J(\mathbf{C}) & \xrightarrow{\sim} & \mathbf{C}^g/\Lambda \end{array}$$

Reconstruct $\hat{h}_{2\vartheta}(\pi(y_K))$ from $\hat{h}(y_K)$ computed using Gross–Zagier.

How to compute $\#Ш(A/\mathbf{Q})_{\mathrm{an}}$ exactly?

- Compute
$$\frac{L(f,1)}{\Omega_f^+} \in \mathbf{Q}(f)$$
using modular symbols.
- If $L(A,1) \neq 0$, this gives $\#Ш(A/\mathbf{Q})_{\mathrm{an}} \in \mathbf{Q}_{>0}$.

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- If $L(A,1) \neq 0$, this gives $\#Ш(A/\mathbf{Q})_{\mathrm{an}} \in \mathbf{Q}_{>0}$.
- If $L(A,1) = 0$:
  - Choose a Heegner field K and compute
$$\frac{L'(A/K,1)}{\mathrm{Reg}_{A/K}\,\Omega_{A/K}} \in \mathbf{Q}_{>0}$$
using Gross–Zagier, and hence compute $\#Ш(A/K)_{\mathrm{an}} \in \mathbf{Q}_{>0}$.
  - Compute $\#Ш(A^K/\mathbf{Q})_{\mathrm{an}} \in \mathbf{Q}_{>0}$.
  - Use $\#Ш(A/K)_{\mathrm{an}} = \#Ш(A/\mathbf{Q})_{\mathrm{an}} \cdot \#Ш(A^K/\mathbf{Q})_{\mathrm{an}}$ up to powers of 2.

How to compute $\#Ш(A/F)_{\mathrm{an}}$ exactly?

Let f be a Hilbert modular form over a totally real field F.

- L-rank 0: Choose twist with $L(f \otimes \chi_D, 1) \neq 0$. Compute $L^{\mathrm{alg}}(f,1)$:

$$L(f,1)\cdot L(f\otimes\chi_D,1) = L(f_K,1) = \frac{2^{[F:\mathbf{Q}]}}{\sqrt{|\mathbf{N}(\mathrm{disc}(K/F))|}}\cdot\|f\|^2\cdot|a(f,D)|^2$$

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$$\frac{L(f,1)}{L(f \otimes \chi_D, 1)} = \frac{c_1(g)^2}{c_{|D|}(g)^2} \sqrt{|\mathbf{N}(D)|}$$

with g a cusp form of weight $\frac{3}{2}$ associated to f.

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  with g a cusp form of weight $\frac{3}{2}$ associated to f.
- L-rank 1 case is reduced to the rank 0 case and the Gross–Zagier–Zhang formula.

Examples in dimension 2

Example: $A = \mathrm{Jac}(X_0(39)/w_{13})$

- $r = r_{\mathrm{an}} = 0$
- $\#\text{Ш}(A/\mathbf{Q})_{\mathrm{an}} = 1$
- $A(\mathbf{Q}) = A(\mathbf{Q})_{\mathrm{tors}} \cong \mathbf{Z}/2 \times \mathbf{Z}/(2 \cdot 7)$

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- $\rho_{\mathfrak{p}}$ is reducible exactly for $\mathfrak{p} = (\sqrt{2})$ and exactly one $\mathfrak{p}\bar{\mathfrak{p}} = 7$.
- $c = 7$
- $I_{\mathbf{Q}(\sqrt{-23})} = 7$ gives $\#\text{Ш}(A/\mathbf{Q})[\mathfrak{p}] = 0$ for $\mathfrak{p} \nmid (\sqrt{2}), 7$.

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- $\rho_{\mathfrak{p}}$ is reducible with

$$0 \to \mathbf{Z}/7 \to A[\mathfrak{p}] \to \mu_7 \to 1$$

non-split exact, and $\mathrm{Sel}_{\mathfrak{p}}(A/\mathbf{Q}) \cong \mathbf{Z}/7 \cong A(\mathbf{Q})[7]$ by descent. Hence $Ш(A/\mathbf{Q})[\mathfrak{p}] = 0$.

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- The $\bar{\mathfrak{p}}$-adic L-function has constant term a unit in $\mathcal{O}_{\bar{\mathfrak{p}}} \simeq \mathbf{Z}_7$, hence the IMC shows $\mathrm{Sel}_{\bar{\mathfrak{p}}}(A/\mathbf{Q}) = 0$.

All Atkin-Lehner quotients of genus 2 of our type (I)

X	r	$\mathcal{O}$	$\#Ш_{an}$	$\rho_\mathfrak{p}$ red.	c	(D, l_D)	$\#Ш$
$X_0(23)$	0	$\sqrt{5}$	1	11_1	11	$(-7, 11)$	11^0
$X_0(29)$	0	$\sqrt{2}$	1	7_1	7	$(-7, 7)$	7^0
$X_0(31)$	0	$\sqrt{5}$	1	$\sqrt{5}$	5	$(-11, 5)$	5^0
$X_0(35)/w_7$	0	$\sqrt{17}$	1	2_1	1	$(-19, 1)$	1
$X_0(39)/w_{13}$	0	$\sqrt{2}$	1	$\sqrt{2}$, 7_1	7	$(-23, 7)$	7^0
$X_0(67)^+$	2	$\sqrt{5}$	1		1	$(-7, 1)$	1
$X_0(73)^+$	2	$\sqrt{5}$	1		1	$(-19, 1)$	1
$X_0(85)^*$	2	$\sqrt{2}$	1	$\sqrt{2}$	1	$(-19, 1)$	1
$X_0(87)/w_{29}$	0	$\sqrt{5}$	1	$\sqrt{5}$	5	$(-23, 5)$	5^0
$X_0(93)^*$	2	$\sqrt{5}$	1		1	$(-11, 1)$	1
$X_0(103)^+$	2	$\sqrt{5}$	1		1	$(-11, 1)$	1
$X_0(107)^+$	2	$\sqrt{5}$	1		1	$(-7, 1)$	1
$X_0(115)^*$	2	$\sqrt{5}$	1		1	$(-11, 1)$	1
$X_0(125)^+$	2	$\sqrt{5}$	1	$\sqrt{5}$	1	$(-11, 1)$	5^0

All Atkin-Lehner quotients of genus 2 of our type (II)

X	r	$\mathcal{O}$	$\#Ш_{\mathrm{an}}$	$\rho_{\mathfrak{p}}$ red.	c	(D, I_D)	$\#Ш$
$X_0(133)^*$	2	$\sqrt{5}$	1		1	$(-31, 1)$	1
$X_0(147)^*$	2	$\sqrt{2}$	1	$\sqrt{2}$, 7_1	1	$(-47, 1)$	7^0
$X_0(161)^*$	2	$\sqrt{5}$	1		1	$(-19, 1)$	1
$X_0(165)^*$	2	$\sqrt{2}$	1	$\sqrt{2}$	1	$(-131, 1)$	1
$X_0(167)^+$	2	$\sqrt{5}$	1		1	$(-15, 1)$	1
$X_0(177)^*$	2	$\sqrt{5}$	1		1	$(-11, 1)$	1
$X_0(191)^+$	2	$\sqrt{5}$	1		1	$(-7, 1)$	1
$X_0(205)^*$	2	$\sqrt{5}$	1		1	$(-31, 1)$	1
$X_0(209)^*$	2	$\sqrt{2}$	1		1	$(-51, 1)$	1
$X_0(213)^*$	2	$\sqrt{5}$	1		1	$(-11, 1)$	1
$X_0(221)^*$	2	$\sqrt{5}$	1		1	$(-35, 1)$	1
$X_0(287)^*$	2	$\sqrt{5}$	1		1	$(-31, 1)$	1
$X_0(299)^*$	2	$\sqrt{5}$	1		1	$(-43, 1)$	1
$X_0(357)^*$	2	$\sqrt{2}$	1		1	$(-47, 1)$	1

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- All $\rho_{\mathfrak{p}}$ are irreducible.

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- $I_K = 2^? \cdot 7$ implies $Ш(J/\mathbf{Q}) \cong (\mathbf{Z}/7)^2 = \#Ш(J/\mathbf{Q})_{\mathrm{an}}$.

Outlook

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- RM abelian threefolds: A generic curve of genus 3 is non-hyperelliptic, so we need an explicit theory of Jacobians and heights.
- Strong BSD over totally real fields.

Thank you!