Descent and étale-Brauer obstructions for 0-cycles (joint with J. Berg)

Symposium on Arithmetic Geometry and its Applications, CIRM

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- **Descent sets**: if $g : Y \to X$ is an X-torsor under some linear algebraic group G over k,

$$X(\mathbf{A}_k)^g := \bigcup_{\sigma \in H^1_{\text{ét}}(k, G)} g^\sigma(Y^\sigma(\mathbf{A}_k)).$$

- **Étale-Brauer set**:

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and we know that $X(\mathbf{A}_k)^{\text{ét,Br}}$ is generally insufficient to explain failure of the Hasse principle.

(Poonen: there is an X with $X(\mathbf{A}_k)^{\text{ét,Br}} \neq \emptyset$ but $X(k) = \emptyset$.)

0-cycles of degree d. The set of 0-cycles of degree of X is the set of $\mathbf{Z}$-formal sums

$$Z_0^d(X) := \left\{ z := \sum_{x \in X \text{ closed pt}} n_x x \;\middle|\; \deg(z) := \sum_{x \in X \text{ closed pt}} n_x [k(x) : k] = d \right\}.$$

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Similarly as with what we do for rational points, we often use obstruction sets cut out from the adelic 0-cycles and containing $Z_0^d(X)$:

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$$\sum_v \sum_{x_v \in X_{k_v}} n_{x_v} \mathrm{inv}_v\big(\mathrm{cores}_{\kappa(x_v)/k_v}(\alpha(x_v))\big) = 0.$$

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Conjecture (Colliot-Thélène)

Let X be nice over k. Then the Brauer-Manin obstruction is the only one to weak approximation for 0-cycles of degree 1 on X.

Rational points vs 0-cycles

k-rational points		0-cycles of degree d
$X(k)$	$\rightsquigarrow$	$Z_0^d(X_k)$
$X(\mathbf{A}_k)$	$\rightsquigarrow$	$Z_0^d(X_{\mathbf{A}_k})$
$X(\mathbf{A}_k)^{\mathrm{Br}\,X}$	$\rightsquigarrow$	$Z_0^d(X_{\mathbf{A}_k})^{\mathrm{Br}}$
$X(\mathbf{A}_k)^g$ for $g : Y \to X$ a torsor under G	$\rightsquigarrow$	???
$X(\mathbf{A}_k)^{\text{ét},\mathrm{Br}}$	$\rightsquigarrow$	???

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Idea: we want a similar partition for the set of global 0-cycles $Z_0^d(X)$, and we can exploit the partition for the rational points to construct it.

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$$T_L := \{[g_L^{-1}(x)] : x \in \mathrm{supp}(z) \text{ and } k(x) = L\} \subset H^1(L, G).$$

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  We note that $\sum_{L \in S} \sum_{\tau \in T_L} \Delta_\tau [L : k] = d$.

Then, using the partition for rational points via the torsor, it is clear that any $x \in P(L, \tau)$ lifts to some L-rational point $y_x \in Y_L^{\tau}$.

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So, by considering all the $x \in \mathrm{supp}(z)$ with a same residue field and "twist", we obtain the "decomposition" into 0-cycles

$$\left(\left(\sum_{x \in P(L,\tau)} n_x y_x\right)_{\tau \in T_L}\right)_{L \in S} \in \prod_{L \in S} \prod_{\tau \in T_L} Z_0^{\Delta_\tau}(Y_L^\tau).$$

These pieces can the be "recombined" into z by taking the pushforwards of the points in all the Y_L^τ's along the natural composition

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Explicitly, the "recombining" map is

$$g_{**}\left(\left(\left(\sum_{x \in P(L,\tau)} n_x y_x\right)_{\tau \in T_L}\right)_{L \in S}\right) = \sum_{L \in S} \sum_{\tau \in T_L} \sum_{x \in P(L,\tau)} n_x (s_L \circ g_L^\tau)_*(y_x).$$

Hence, any $z \in Z_0^d(X)$ comes from

$$Z_0^d(X)^g := g_{**}\left(\coprod_S \coprod_{(T_L)_{L\in S}} \coprod_{\substack{\left((\Delta_\tau)_{\tau\in T_L}\right)_{L\in S}:\\ \sum_{L\in S}\sum_{\tau\in T_L}[L:k]\Delta_\tau = d}} \prod_{L\in S}\prod_{\tau\in T_L} Z_0^{\Delta_\tau}(Y_L^\tau)\right)$$

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(In fact, if $d = 1$ and if we consider effective 0-cycles and $S = \{k\}$ only, then we recover $X(k) = \bigcup_\tau g^\tau(Y^\tau(k))$.)

So, we define

- the g-**descent set of degree** d to be

$$Z_0^d(X_{\mathbf{A}_k})^g := g_{**}^{ad}\left(\coprod_{S}\coprod_{(T_L)_{L\in S}}\coprod_{\substack{\left((\Delta_\tau)_{\tau\in T_L}\right)_{L\in S}:\\ \sum_{L\in S}\sum_{\tau\in T_L}[L:k]\Delta_\tau=d}}\prod_{L\in S}\prod_{\tau\in T_L} Z_0^{\Delta_\tau}(Y_{\mathbf{A}_L}^{\tau})\right)$$

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Question: Do we always have that

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The conjecture by Colliot-Thélène says YES.

Some results: 0-cycles on Enriques surfaces

Recall that if X is an Enriques surface over k, there is always a K3 covering $f : Y \to X$, which is a $\mathbf{Z}/2\mathbf{Z}$-torsor over X with Y a K3 surface.

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For K3 surfaces, the arithmetic behaviour of rational points is conjecturally explained by the Brauer-Manin obstruction.

Conjecture (Skorobogatov)

Let Y be a K3 surface over k. Then $\overline{Y(k)} = Y(\mathbf{A}_k)^{\mathrm{Br}}$.

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Then: *for any positive integer n, if $(z_v)_v \in Z_0^d(X_{\mathbf{A}_k})^{f,\mathrm{Br}}$ then there exists a global 0-cycle $z_n \in Z_0^d(X)$ such that z_n and $(z_v)_v$ have the same image in $CH_0(X_{k_v})/n$ for all $v \in \Omega_k$.*

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Note: the proof uses the following result by Ieronymou: under Skorobogatov's conjecture, the Brauer-Manin obstruction is the only one for weak approximation (in the Chow groups sense) for 0-cycles of any degree d on a K3 surface.

Some results: 0-cycles on varieties admitting a universal torsor

A classical result about universal torsors is the following.

Theorem

Let X be a variety over k with $\overline{k}[X]^\times = \overline{k}^\times$ and $\operatorname{Pic} \overline{X}$ finitely generated as a $\mathbf{Z}$-module. Assume that $g : W \to X$ is a universal torsor for X. Then $X(\mathbf{A}_k)^g = X(\mathbf{A}_k)^{\mathrm{Br}_1}$.

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(Here $\mathrm{Br}_1(X) := \ker(\operatorname{Br} X \to \operatorname{Br} \overline{X})$.)

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1. $\mathbf{Z}_0^d(X_{\mathbf{A}_k})^g \neq \emptyset$ *implies* $\mathbf{Z}_0^d(X_{\mathbf{A}_k})^{\mathrm{Br}_1} \neq \emptyset$;

Extending to our setting a strategy of Liang (i.e. that of relating the arithmetic of 0-cycles to that of rational points over finite extensions of k), we can prove an analogue result for 0-cycles.

Theorem (B.-Berg)

Let X be a smooth, proper, geometrically integral variety over k. Suppose that a universal torsor $g : W \to X$ exists.

Then: *for any integer $d \in \mathbf{Z}$, for any positive integer n, and for any finite subset $S \subset \Omega_k$ of places of k, we have that*

1. $\mathbf{Z}_0^d(X_{\mathbf{A}_k})^g \neq \emptyset$ *implies* $\mathbf{Z}_0^d(X_{\mathbf{A}_k})^{\mathrm{Br}_1} \neq \emptyset$;
2. *if, moreover,* $\mathrm{Br}_1(X)/\mathrm{Br}_0(X)$ *is finite, then* $(z_v)_v \in \mathbf{Z}_0^d(X_{\mathbf{A}_k})^{\mathrm{Br}_1}$ *implies that there exists some* $(u_v)_v \in \mathbf{Z}_0^d(X_{\mathbf{A}_k})^g$ *such that* z_v *and* u_v *have the same image in* $CH_0(X_{k_v})/n$ *for all* $v \in S$.

Some results: 0-cycles on varieties with a torsor under a torus

In the rational points setting, Harpaz and Wittenberg have proved the following result.

Theorem (Harpaz-Wittenberg)

Let X be a smooth, geometrically integral variety over k. Let $f : Y \to X$ be a torsor under a k-torus T. Let $A \subset \operatorname{Br} X$ be the inverse image of $\operatorname{Br}_{nr}(Y) \subset \operatorname{Br} Y$ under $f^* : \operatorname{Br} X \to \operatorname{Br} Y$. Then

$$X(\mathbf{A}_k)^A \subset \bigcup_{\sigma \in H^1(k,T)} f^\sigma(Y^\sigma(\mathbf{A}_k)^{\operatorname{Br}_{nr}}).$$

Using again Liang's strategy, we can prove the following analogue for 0-cycles.

Theorem (B.-Berg)

Let X be a smooth, proper, geometrically integral variety over k. Let $f : Y \to X$ be a torsor under a k-torus T.

Using again Liang's strategy, we can prove the following analogue for 0-cycles.

Theorem (B.-Berg)

Let X be a smooth, proper, geometrically integral variety over k. Let $f : Y \to X$ be a torsor under a k-torus T.

Assume that $\operatorname{Br} X / \operatorname{Br}_0 X$ is finite and that there is some finite extension F/k such that $\operatorname{res}_{L/k} : \operatorname{Br} X / \operatorname{Br}_0 X \to \operatorname{Br}(X_L) / \operatorname{Br}_0(X_L)$ is surjective for all finite extensions L/k linearly disjoint from F over k.

Using again Liang's strategy, we can prove the following analogue for 0-cycles.

Theorem (B.-Berg)

Let X be a smooth, proper, geometrically integral variety over k. Let $f : Y \to X$ be a torsor under a k-torus T.

Assume that $\operatorname{Br} X / \operatorname{Br}_0 X$ is finite and that there is some finite extension F/k such that $\operatorname{res}_{L/k} : \operatorname{Br} X / \operatorname{Br}_0 X \to \operatorname{Br}(X_L)/\operatorname{Br}_0(X_L)$ is surjective for all finite extensions L/k linearly disjoint from F over k.

Then: *for any integer $d \in \mathbf{Z}$, for any positive integer n, and for any finite subset $S \subset \Omega_k$ of places of k, we have that $(z_v)_v \in \mathbf{Z}_0^d(X_{\mathbf{A}_k})^{\operatorname{Br}}$ implies that there exists some $(u_v)_v \in \mathbf{Z}_0^d(X_{\mathbf{A}_k})^{f,\operatorname{Br}_{nr}}$ such that z_v and u_v have the same image in $CH_0(X_{k_v})/n$ for all $v \in S$.*

Thank you for your attention!