Counting integral points on symmetric varieties, and applications to arithmetic statistics

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SAGA 2023
CIRM

February 9th, 2023

Motivation: 2-class groups in families of cubic fields

Theorem (Bhargava, 2005)

When totally real (resp., complex) cubic fields K are ordered by the absolute values of their discriminants, the average size of $\mathrm{Cl}(\mathcal{O}_K)[2]$ is $5/4$ (resp., $3/2$).

- Delone–Faddeev: $f \in \mathbb{Z}[x, y] \rightsquigarrow R_f := H^0(\mathrm{Proj}\,\mathbb{Z}[x, y]/(f))$ defines bijection between $\mathrm{GL}_2(\mathbb{Z})$-orbits of binary cubics and cubic rings/iso.

Theorem (Ho–Shankar–Varma, 2018)

When binary cubics $f \in \mathbb{Z}[x, y]$ such that R_f is the ring of integers of a totally real (resp., complex) cubic field are ordered by the maximum of the sizes of their coefficients, the average size of $\mathrm{Cl}(R_f)[2]$ is $5/4$ (resp., $3/2$).

- Remarkably, averages remain unchanged under imposition of general infinite sets of local conditions

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- What happens to averages upon imposing global conditions?
- Consider $U_a \subset \mathbb{Z}[x, y]$ the subset of binary cubics with first coeff. a (e.g., $a = 1 \implies R_f = \mathbb{Z}[x]/(f(x, 1))$ is monogenic)

Theorem (Bhargava–Hanke–Shankar, 2019)

Let $a \in \mathbb{Z} \setminus \{0\}$. When binary cubics $f \in U_a$ such that R_f is the ring of integers of a totally real (resp., complex) cubic field are ordered by height, the average size of $\mathrm{Cl}(R_f)[2]$ is

$$5/4 + 1/4\sigma(k_a) \quad (\textit{resp., } 3/2 + 1/2\sigma(k_a))$$

where σ = divisor function, k_a = product of odd-mult. prime factors of a.

- Height of $ax^3 + bx^2y + cxy^2 + dy^3$ is $\max\{|b|, |ac|^{1/2}, |a^2d|^{1/3}\}$
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Main Result

- What about fixing two coefficients? Consider $U_{a,d}(\mathbb{Z}) \subset \mathbb{Z}[x, y]$ the subset of binary cubics with first coeff. a and last coeff. d

Theorem (in progress, Setayesh, Shankar, Siad, S., 2023)

Consider binary cubics $f \in U_{1,1}(\mathbb{Z})$ such that R_f is the ring of integers of a totally real cubic field in which 2 is inert. When such cubics are ordered by the sizes of their coefficients, the average size of $\mathrm{Cl}(R_f)[2]$ is $\leq^\star 2$.

- Imposing monogenicity causes average 2-torsion in class group to increase; double-monogenicity makes it increase even further!
- Assumed $a = d = 1$ and certain splitting conditions at ∞, 2 for simplicity, can prove a more general theorem, without these conditions
- Bound (and conditional equality) remains unchanged under general infinite sets of local conditions away from $p \mid 2ad$

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Parametrization of 2-torsion classes of cubic fields

- Parametrize arithmetic objects of interest in terms of integral orbits of a representation $G \curvearrowright W$
- Let $W = \{\text{pairs } (A, B) \text{ of ternary quadratic forms}\}$ and $G = \mathrm{SL}_3$; then $\mathrm{SL}_3 \curvearrowright W$ via linear change-of-variable, with ring of invariants generated over $\mathbb{Z}$ by the coefficients of $4 \times \det(xA - yB)$

Theorem (Bhargava, 2005)

Let $f \in \mathbb{Z}[x, y]$ be a binary cubic form s. t. R_f is the ring of integers of a number field, and let $W^+(\mathbb{R}) := \{(A, B) \in W(\mathbb{R}) : A(\mathbb{R}) \cap B(\mathbb{R}) \neq \varnothing\}$. Then the elements of $\mathrm{Cl}(R_f)[2]^\vee \smallsetminus \{1\}$ are in natural bijection with the $\mathrm{SL}_3(\mathbb{Z})$-orbits of pairs $(A, B) \in W(\mathbb{Z}) \cap W^+(\mathbb{R})$ such that $A(\mathbb{Q}) \cap B(\mathbb{Q}) = \varnothing$ and $4 \times \det(xA - yB) = f(x, y)$.

- For $a, d \in \mathbb{Z} \smallsetminus \{0\}$, let $W_{a,d} = \{(A, B) \in W : \det A = a, \det B = d\}$; suffices to get asymptotics for number of $\mathrm{SL}_3(\mathbb{Z})$-orbits on $W_{a,d}(\mathbb{Z}) \cap W^+(\mathbb{R})$ satisfying sets of local conditions

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Parametrization of 2-torsion classes of cubic fields

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Overview of the counting argument

- View integral orbits as lattice pts. in a fund. set $\mathcal{F}$ for action of $\mathrm{SL}_3(\mathbb{Z})$ on $W^+(\mathbb{R})$, lying on hypersurface $W_{a,d}(\mathbb{R})$
- Explicitly, take $\mathcal{F} = \mathcal{F}_1\mathcal{F}_2$, where $\mathcal{F}_1 = \mathrm{SL}_3(\mathbb{Z})\backslash\,\mathrm{SL}_3(\mathbb{R})$ and $\mathcal{F}_2 = \mathrm{SL}_3(\mathbb{R})\backslash W^+(\mathbb{R})$
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- $\mathcal{F}_1$ consists of "main body" $\mathcal{F}_1^{\text{main}}$ and "cusp" $\mathcal{F}_1^{\text{cusp}}$:
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Symmetric varieties

- $V :=$ ternary quadratic forms, $V_a := \{A \in V : \det A = a/4\}$.
- Fix $A \in V_a(\mathbb{Z})$, and for a $\mathbb{Z}$-algebra R let $S_A(R) := \mathrm{SL}_3(R)/\mathrm{SO}_A(R)$. The map $\pi_A \colon \mathrm{SL}_3(\mathbb{R}) \to V_a(\mathbb{R})$ defined by $g \mapsto gAg^T$ induces an injection from $S_A(\mathbb{R}) \hookrightarrow V_a(\mathbb{R})$; moreover, we have

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Theorem (Eskin–McMullen, 1993)

Let $\mathcal{B} \subset S_A(\mathbb{R})$ be a cpt subset of nonzero volume w.r.t. Haar measure on $S_A(\mathbb{R}) = \mathrm{SL}_3(\mathbb{R})/\mathrm{SO}_A(\mathbb{R})$, and assume $\mathcal{B}$ is "well-rounded." Then

$$\#(S_A(\mathbb{Z}) \cap X\mathcal{B}) \sim \frac{\mathrm{Vol}(\mathrm{SO}_A(\mathbb{Z})\backslash \mathrm{SO}_A(\mathbb{R}))}{\mathrm{Vol}(\mathrm{SL}_3(\mathbb{Z})\backslash \mathrm{SL}_3(\mathbb{R}))} \times \mathrm{Vol}(X\mathcal{B}).$$

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Asymptotics for integral orbits

Theorem (in progress, Setayesh, Shankar, Siad, S., 2023)

For any $A \in V_a(\mathbb{Z})$ and $B \in V_d(\mathbb{Z})$, we have

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Application to 2-Selmer groups of elliptic curves

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When elliptic curves in the family $y^2 = x^3 + ax^2 + bx + 1$ with full real 2-torsion and squarefree discriminant are ordered by $\max\{|a|, |b|\}$, the average size of the 2-Selmer group is $\leq^\star 6$.

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Related work

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- In general, seems like one always obtains asymptotic orbit count of the following form:

  [volume of invts] $\times$ [Tamagawa number(s)] $\times$ [product of local masses]

- Question: For what types of varieties do we get such an asymptotic? (e.g., affine space, quadrics, symmetric varieties)

Related work

- Alpöge: count integral orbits lying on quadric hypersurfaces using circle method, with applications to studying Mordell curves:
  - Alpöge–Bhargava: determine average size of the 2-Selmer group in the family $y^2 = x^3 + k$;
  - Alpöge–Bhargava–Shnidman: positive proportions of cubic and quartic number fields fail to be monogenic despite no local obstructions;
  - Alpöge–Bhargava–Shnidman: positive proportion of integers can(not) be written as sum of two rational cubes
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Thank You!!