On the 2-Selmer group of hyperelliptic curves.

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Descent on elliptic curves

Let E/K be an elliptic curve and let n be a positive integer. The short exact sequence of Gal_K-modules

$$1 \longrightarrow E(\overline{K})[n] \longrightarrow E(\overline{K}) \xrightarrow{\times n} E(\overline{K}) \longrightarrow 1$$

induces a long exact sequence

$$1 \longrightarrow E(K)/nE(K) \xrightarrow{\delta} \mathrm{H}^1(\mathrm{Gal}_K, E[n]) \longrightarrow \mathrm{H}^1(\mathrm{Gal}_K, E) \rightarrow$$

Then to compute/bound $E(K)$ it is enough to compute:

- The finite group $E(K)[n]$,
- The cohomology group $\mathrm{H}^1(\mathrm{Gal}_K, E[n])$.

⚠ The group $\mathrm{H}^1(\mathrm{Gal}_K, E[n])$ is not finite (we need to impose some ramification condition)

An easy case

Suppose that $E(K)[n] = E[n]$. In particular, $\zeta_n \in K$. Then

- $\mathrm{H}^1(\mathrm{Gal}_K, E[n]) \simeq \mathrm{Hom}(\mathrm{Gal}_K, \mathbb{Z}/n) \times \mathrm{Hom}(\mathrm{Gal}_K, \mathbb{Z}/n)$.
- $\mathrm{Hom}(\mathrm{Gal}_K, \mathbb{Z}/n) \leftrightsquigarrow$ cyclic degree n field extensions of K,
- By Kummer's theory, cyclic degree n field extensions of K correspond to elements in $K^\times/(K^\times)^n$.

Let S be the set of primes of bad reduction of E together with the primes dividing n and let

$$K(S,n) = \{b \in K^\times \,:\, n \mid v_{\mathfrak{p}}(b) \text{ for all } \mathfrak{p} \notin S\}/(K^\times)^n.$$

Then one can actually prove that $E(K)/nE(K) \hookrightarrow K(S,n) \times K(S,n)$.
⚠ It is a hard problem to determine which elements of $K(S,n)$ actually lie in the image

Improvements

- When $E(K)[n] \subsetneq E[n]$, one can look at $K(E[n])$ and do a similar computation. Can we do better?
- (Cassels) If
$$E : y^2 = p(x)$$
and $n = 2$ it is enough to look at $A_K = K[x]/(p(x))$.
- Under some hypothesis, we can remove the bad primes and 2 from the ramification set.
- Instead of looking at $\mathrm{H}^1(\mathrm{Gal}_{K,S}, E[n])$, work with $\mathrm{Sel}_n(E)$ (the n-th Selmer group).

Hyperelliptic curves

Let K be a number field with odd class number and let

$$\mathscr{C} : y^2 = p(x),$$

where $p(x) \in \mathcal{O}[x]$ monic, of odd degree and a_{d-1} even. Let J denote its Jacobian. Let

$$A_K := K[x]/(p(x)).$$

Theorem

Under the following hypothesis

$$\begin{aligned} \dim_{\mathbb{F}_2} \mathrm{Cl}_*(A_K, \mathscr{C})[2] - \sum_{v|2} \left(r_v - 1 - \dim_{\mathbb{F}_2}(W_v)\right) \quad &\leq \\ \leq \quad \dim_{\mathbb{F}_2} \mathrm{Sel}_2(J) \quad &\leq \quad \dim_{\mathbb{F}_2} \mathrm{Cl}_*(A_K, \mathscr{C})[2] + g\,[K:\mathbb{Q}]\,. \end{aligned}$$

Formula:

$$\dim_{\mathbb{F}_2} \mathrm{Cl}_*(A_K,\mathscr{C})[2] - \sum_{v|2}\left(r_v - 1 - \dim_{\mathbb{F}_2}(W_v)\right) \quad \leq$$

$$\leq \quad \dim_{\mathbb{F}_2} \mathrm{Sel}_2(J) \quad \leq \quad \dim_{\mathbb{F}_2} \mathrm{Cl}_*(A_K,\mathscr{C})[2] + g\,[K:\mathbb{Q}]. \quad (1)$$

Where for $v \mid 2$,

- r_v is the number of factors of $p(x)$ modulo v.
- If $A_{K_v} \simeq K_1 \times \cdots \times K_r$, with $K_i = K_v[x_i]/(p_i(x_i))$. Then

$$W_v := \langle \mathrm{Tr}_{k_i/k}(\overline{x}_i) \ : \ i = 1, ..., r \rangle \subset k.$$

- g is the genus of $\mathscr{C} = \frac{d-1}{2}$.
- $\mathrm{Cl}_*(A_K,\mathscr{C})[2]$ is the two torsion of the class group with the following hypothesis at infinity

Hypotheses and ideas

- The group $\mathrm{H}^1(\mathrm{Gal}_K, J[2])$ is isomorphic to $(A_K^\times/(A_K^\times)^2)_\square$.
- For bounding $\mathrm{Sel}_2(J)$, compute for each place v the image of

$$\delta_v : J(K_v)/2J(K_v) \hookrightarrow (A_{K_v}^\times/(A_{K_v}^\times)^2)_\square$$

- $p(x)$ satisfies condition $(\dagger)$ if one of the following holds:
  - $(\dagger.\mathrm{i})$ The K-algebra A_K is a field extension of K.
  - $(\dagger.\mathrm{ii})$ The ring of integers $A_\mathcal{O}$ equals $\mathcal{O}[x]/(p(x))$.
  - $(\dagger.\mathrm{iii})$ The residual characteristic of K is odd and the order of the component group $[J(K) : J^0(K)]$ is odd.

Theorem

If the polynomial $p(x)$ satisfies $(\dagger)$ then $\mathrm{Im}(\delta_v) \subset (A_{\mathcal{O}_v}^\times/(A_{\mathcal{O}_v}^\times)^2)_\square$.

If $v \nmid 2$ we get equality. This implies no ramification at v

Places dividing 2 and ∞

- When $v \mid 2$, construct a set W_v such that

$$W_v \subset \mathrm{Im}(\delta_v) \subset (A^{\times}_{\mathcal{O}_v}/(A^{\times}_{\mathcal{O}_v})^2)_{\square}$$

  and compute $\#W_v/(A^{\times}_{\mathcal{O}_v})^2$ (need to construct local points).
- A similar construction at infinity.
- Define "class groups" $C_W(\mathscr{C}), \tilde{C}(\mathscr{C})$ and prove that

$$C_W(\mathscr{C}) \subset \mathrm{Sel}_2(J) \subset \tilde{C}(\mathscr{C})$$

- Relate $\tilde{C}(\mathscr{C})$ to $\mathrm{Cl}_*(A_K, \mathscr{C})[2]$ and compute $[\tilde{C}(\mathscr{C}) : C_W(\mathscr{C})]$.
- **Remark:** have many examples that prove that the lower and the upper bound are optimal.

Applications I: quadratic twists

If $a \in \mathcal{O}$ the twist of $\mathscr{C}$ by a equals

$$\mathscr{C}_a : y^2 = a^d p(x/d).$$

- If $p(x)$ is irreducible and all prime ideals dividing a are inert or totally ramified in A_K/K, then if $\mathscr{C}$ satisfies † so does $\mathscr{C}_a$. ▸ H

Theorem

Suppose that the Galois group of the Galois closure of A_K over K contains an element of order d. Then among all quadratic twists by principal prime ideals, there exists a subset of positive density $\mathcal{S}$ such that the abelian varieties $J(\mathscr{C}_a)$ have the same 2-Selmer group for all $a \in \mathcal{S}$.

Application II: a particular family

Let a be a non-zero integer, and consider the genus 2 hyperelliptic curve

$$\mathscr{C}_a : y^2 = x^5 - ax.$$

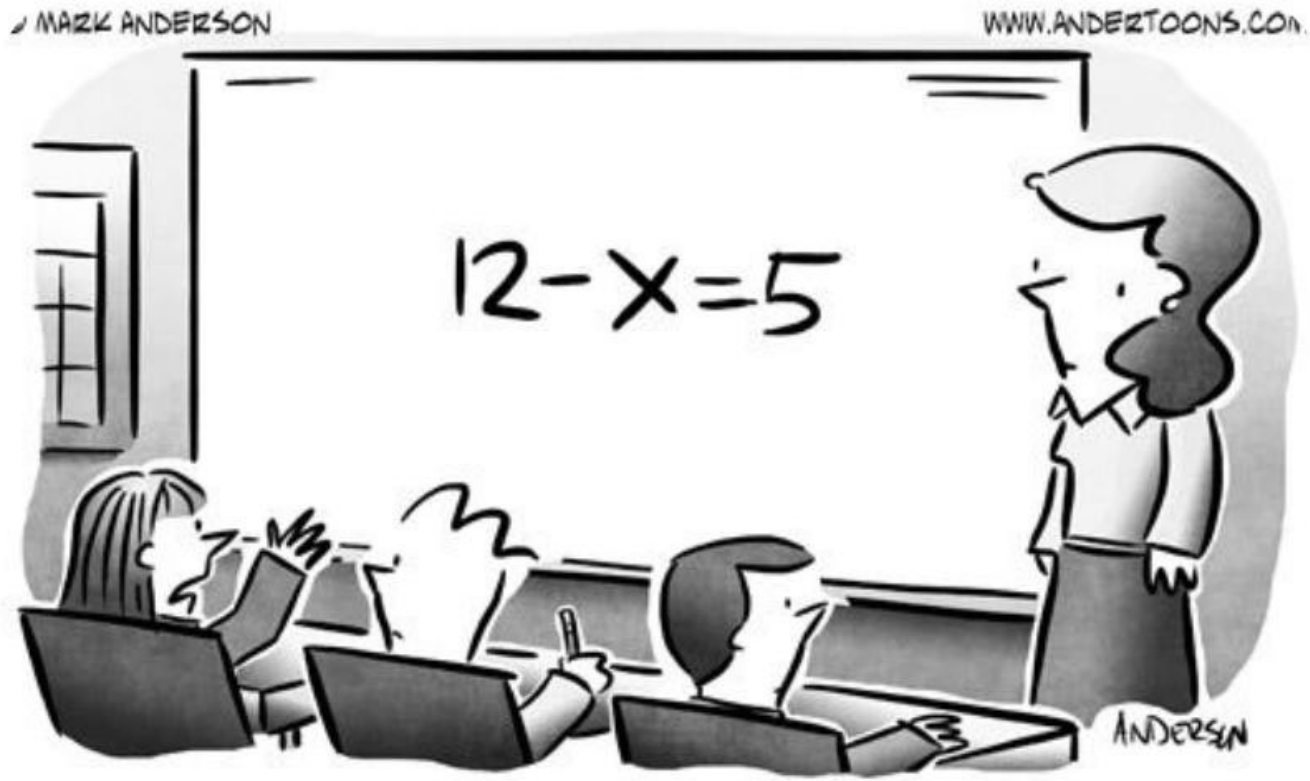
Theorem

Assuming the parity conjecture, $J(\mathscr{C}_p)$ has rank 1 for all primes $p \equiv 3 \pmod 8$.

Proof.

- If $p \equiv 3 \pmod 4$ then $\mathrm{Cl}(\mathbb{Q}[\sqrt[4]{p}])$ is odd.
- Applying our result to $\mathscr{C}_p$, $0 \leq \dim_{\mathbb{F}_2} \mathrm{Sel}_2(J(\mathscr{C}_p)) \leq 2$.
- Since $J(\mathscr{C}(p))$ has a 2-torsion point, $1 \leq \dim_{\mathbb{F}_2} \mathrm{Sel}_2(J(\mathscr{C}_p)) \leq 2$.
- Compute the root number (use the curve has CM).

Thank you for coming



"This time X is 7. Last time it was 4. And the time before that it was 12. Maybe X should take some time to reflect and decide for itself what it really wants to be."