The congruence ideal associated to p-adic families of Yoshida lifts

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Symposium on Arithmetic Geometry and its Applications (SAGA)

February 2023

An ongoing work, joint with

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Motivation: Herbrand–Ribet theorem / Irregular primes

- Let p denote an odd prime.
- Let A denote the p-primary part of the class group of $\mathbb{Q}(\zeta_p)$.

$$\omega : \mathrm{Gal}\left(\mathbb{Q}(\zeta_p)/\mathbb{Q}\right) \cong (\mathbb{Z}/p\mathbb{Z})^\times \hookrightarrow \mathbb{Z}_p^\times.$$

$$\mathrm{A} \cong \bigoplus_{j=0}^{p-2} A_j, \qquad A_j := A^{\omega^j}.$$

Theorem (Herbrand–Ribet)

Let k be an even integer between 2 and $p-1$. The following statements are equivalent.

1. *p divides the numerator of $\zeta(1-k)$.*
2. $A_{p-k} \neq 0$.

Ribet's method: congruences involving Eisenstein series

$$
\begin{tikzcd}
\boxed{A_{p-k} \neq 0} \arrow[rr, bend left, "\text{Herbrand}"] & & \boxed{p \text{ divides } \zeta(1-k)} \arrow[dl, bend left, "\text{Ribet}"] \\
& \boxed{\text{Congruences between Eisenstein series and cuspforms}} \arrow[ul, bend left, "\text{Ribet}"] &
\end{tikzcd}
$$

The idea for the converse can also be traced to earlier (unpublished) calculations of Greenberg, Monsky.

An example

Take $p = 691$ and $k = 12$.

$$\zeta(1-12) = \frac{691}{32760}.$$

Then, $A_{691-12} \cong \mathbb{Z}/691\mathbb{Z}$.

The q-expansion $E_{12}(q)$

$\frac{691}{65520} + q + 2049q^2 + 177148q^3 + 4196353q^4 + 48828126q^5 + O(q^6)$.

The q-expansion of $\tau_{12}(q)$:

$0 + q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 + O(q^6)$.

Their difference divided by 691:

$\frac{1}{65520} + 3q^2 + 256q^3 + 6075q^4 + 70656q^5 + O(q^6)$.

An upgrade to p-adic families: Iwasawa Main conjecture

Theorem (Mazur–Wiles, Ohta, Rubin)

Fix an odd integer $3 \leq i \leq p-2$. We have an equality of ideals in Λ:

$$\mathrm{Char}(X_i) = (\theta_i).$$

- The Iwasawa algebra Λ is a completed group ring

$$\varprojlim_n \mathbb{Z}_p \left[\mathrm{Gal}\left(\mathbb{Q}(\mu_{p^n})/\mathbb{Q}(\mu_p)\right)\right].$$

It is isomorphic to a power series ring $\mathbb{Z}_p[[x]]$.

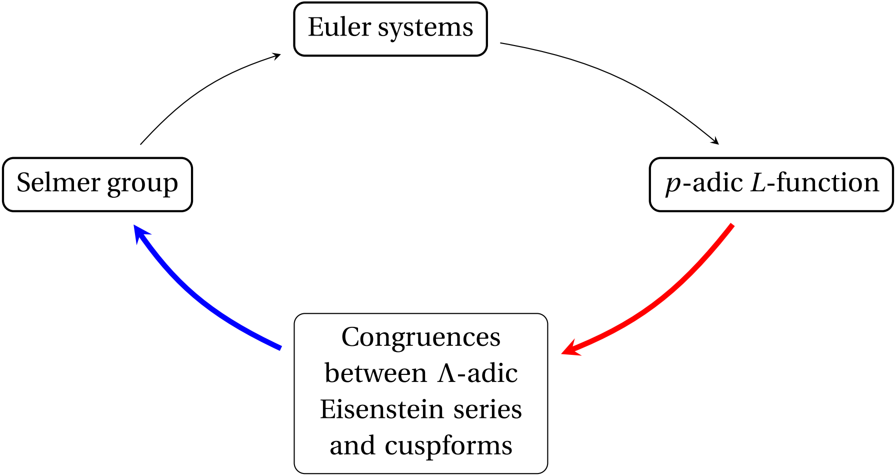
- The Iwasawa module X_i is the ω^i-eigencomponent of the class group of $\mathbb{Q}(\mu_{p^\infty})$.

It is a torsion-module over Λ.

- θ_i is the Kubota–Leopoldt p-adic L-function.

It is an element in Λ.

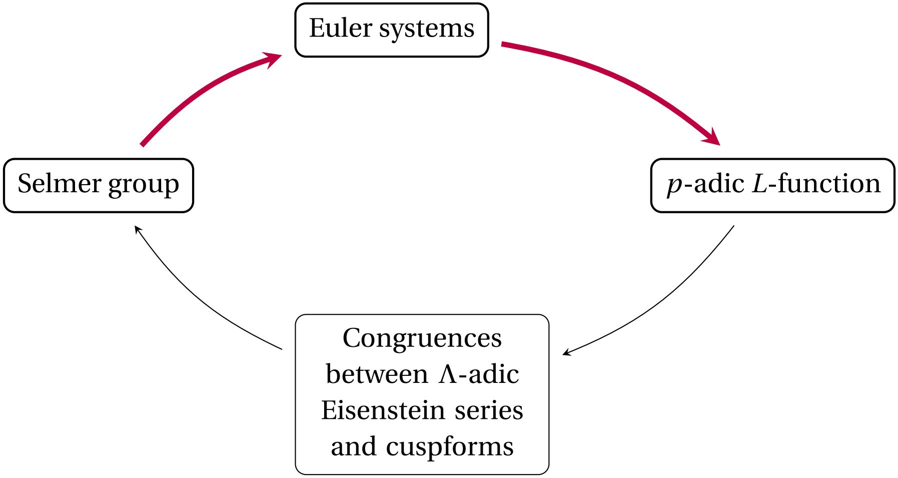
Divisibility afforded by the method of congruences



Theorem (Mazur–Wiles, Ohta)

$$\mathrm{Char}(X_i) \subset (\theta_i).$$

The divisibility afforded by the Euler system method



Theorem (Rubin, Kolvyagin, Thaine)

$$\mathrm{Char}(X_i) \supset (\theta_i).$$

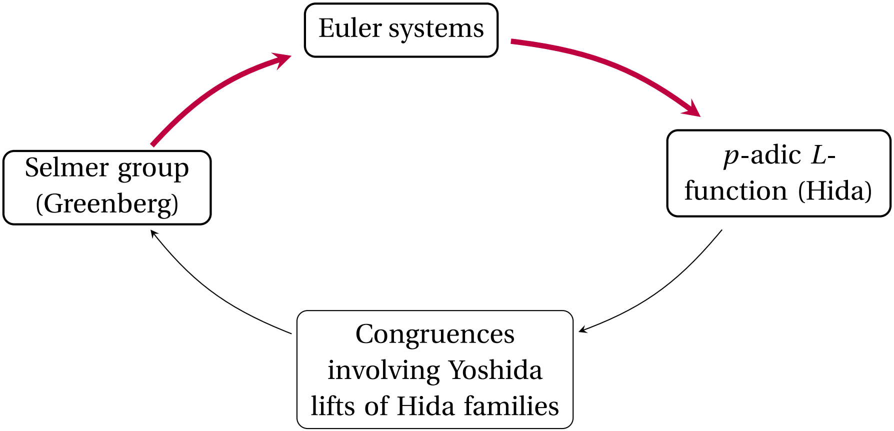
Main conjecture: Rankin–Selberg product of Hida families

- In general, both methods involving *Euler systems* and *Congruences* might be required to prove main conjectures since generalisations of the analytic class number formula might not be available.

- The method of congruences introduces a third ideal and leads to the opposite inclusion:

$$\mathrm{Char}(\mathrm{Sel}) \overset{?}{\subset} \text{cong ideal} \overset{?}{\subset} (\theta).$$

Main conjecture: Rankin–Selberg product of Hida families



Theorem (Lei–Loeffler–Zerbes, Kings–Loeffler–Zerbes)

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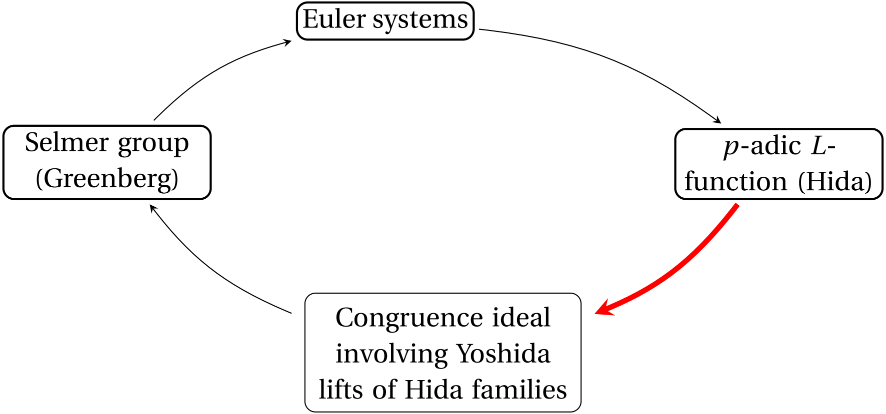
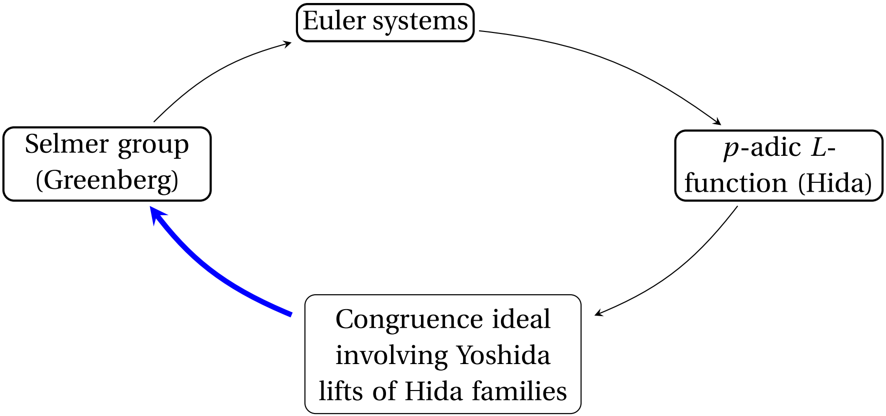
Main conjecture: Rankin–Selberg product of Hida families



Theorem (ongoing work of Hsieh-Liu)

$$\text{cong ideal} \subset (\theta).$$

Main result



What we prove is

Theorem (Hsieh–P.)

$$\mathrm{Char}(\mathrm{Sel}) \subset \text{cong ideal.}$$

Previous works on Yoshida lifts

- Here's an incomplete list of authors who have studied the Yoshida lifts of two cusp forms:
  - Yoshida
  - Böcherer–Schulze-Pillot
  - Agarwal–Klosin
  - Böcherer–Dummigan–Schulze-Pillot
  - Roberts
  - Saha
  - Saha–Schmidt
  - Hsieh–Namikawa
- Agarwal–Klosin and Böcherer–Dummigan–Schulze-Pillot used congruences with Yoshida lifts towards Bloch–Kato conjectures for Rankin–Selberg product of cuspforms.

Formalizing congruences (Hida, Doi–Hida, Ribet)

- $\mathbb{T}$: the local component of the Hecke algebra, corresponding to E_k, acting on $M_k(\mathrm{SL}_2(\mathbb{Z}))$.

$$\begin{aligned} \phi_{E_k}: \mathbb{T} &\twoheadrightarrow \mathbb{Z}_p, \\ T_l &\to 1 + l^{k-1}. \end{aligned}$$

- $\mathbb{T}_{\mathrm{cusp}}$: the max. quotient of $\mathbb{T}$ acting faithfully on $S_k(\mathrm{SL}_2(\mathbb{Z}))$.

$$\mathbb{T} \twoheadrightarrow \mathbb{T}_{\mathrm{cusp}}$$

- The image of $\ker(\phi_{E_k})$ in $\mathbb{T}_{\mathrm{cusp}}$ is called the Eisenstein ideal.

$$\begin{tikzcd} \mathbb{T} \arrow[r, "\phi_{E_k}"] \arrow[d] & \mathbb{Z}_p \arrow[d] \\ \mathbb{T}_{\mathrm{cusp}} \arrow[r] & \star \end{tikzcd}$$

$$\frac{\mathbb{T}_{\mathrm{cusp}}}{\mathrm{Eis}} \cong \frac{\mathbb{Z}_p}{(\text{cong ideal})}.$$

Λ-adic congruence ideals

$$\begin{tikzcd} \mathbf{T} \arrow[r, "\phi"] \arrow[d] & \Lambda \arrow[d] \\ \mathbf{T}_\perp \arrow[r] & \star \end{tikzcd} \qquad\qquad \frac{\mathbf{T}_\perp}{?} \cong \frac{\Lambda}{\text{(cong ideal)}}.$$

- Emerton (level 1) and Ohta (general level) studied the map ϕ associated to the Λ-adic Eisenstein series so that $\perp =$ cusp and $? =$ Eis.

- **Hida–Tilouine** have studied the map ϕ associated to Λ-adic CM forms to prove one divisibility towards **anti-cyclotomic main conjectures**. It is this work of Hida–Tilouine that is our main source of inspiration.

- Our main goal is to generalize the congruence ideal method of Hida–Tilouine to the case of (tempered) endoscopic congruences for GSp_4.

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Ordinary modular forms

- f_{new} be a cuspidal eigen newform in $S_k(\Gamma_1(N))$ with weight ≥ 2.
- We assume throughout that p does not divide N.
- We will also assume that f_{new} is p-ordinary. That is, the p-adic valuation of the Hecke polynomial at p equals

$$0, \qquad k-1.$$

- Let f be the ordinary p-stabilization of f_{new}. This is a cuspform in $S_k(\Gamma_1(Np))$.
- f is an eigenform for the Hecke operators T_l and the diamond operators S_l for l not divinding N along with the U_p operator.
- Let T denote the local component (corresponding to f) of the p-adic Hecke algebra generated by these operators.

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Hida families (GL_2)

Suppose F and G are two Hida families passing through two ordinary p-stabilizations f_0 and g_0.

$$\phi : \mathbf{T} \to \mathbf{I} \tag{1}$$

- $\mathbf{T}$, $\mathbf{I}$ are finitely generated over a subring over $\mathbb{Z}_p[[x]]$.
- $\mathbf{T}$: Λ-adic Hecke algebra generated by the Hecke operators T_l and diamond operators S_l for primes l not divinding N and the U_p operator acting on the space of Λ-adic cuspforms.
- $\mathbf{T}$ is reduced.
- $\mathbf{I}$ is the normalization of $\mathbf{T}/\eta$, for some minimal prime ideal η.
- There exists a dense set of height one prime ideals $\mathfrak{p}_k$ in $\mathbf{T}$ (with $k \geq 2$) containing η and $(1+x)^m - (1+p)^{mp^{k-1}}$, for some $m \geq 1$, such that
$$\mathbf{T} \xrightarrow{\phi} \mathbf{I} \to \mathbf{T}/\mathfrak{p}_k \hookrightarrow \overline{\mathbb{Z}}_p$$
corresponds to the Hecke eigensystem associated to a classical p-ordinary form of weight k. One of these prime ideals correspond to f_0 (and g_0 respectively).

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Rankin–Selberg main conjectures

Theorem (Hida)

There exists a continuous Galois representation

$$\rho_F : \mathrm{Gal}\,(\mathbb{Q}_S/\mathbb{Q}) \to \mathrm{GL}_2(\mathrm{Frac}(\mathbf{I}_F))$$

such that specialization of ρ_F at a classical prime is isomorphic to Shimura–Deligne Galois representation.

We are interested in studying the Iwasawa main conjecture associated to the Galois representation given by the acton of $\mathrm{Gal}\,(\mathbb{Q}_S/\mathbb{Q})$ on

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Rigidity: do properties propagate along p-adic families?

1. Suppose you have a Hida family F passing through an Eisenstein series with weight ≥ 2. Are all the classical specializations (with weight ≥ 2) of the Hida family F Eisenstein? (Yes due to Hida)
2. Suppose you have a Hida family F passing through a CM form f with weight ≥ 2. Are all the classical specializations (with weight ≥ 2) of the Hida family F CM forms? (Yes due to Hida)

- Hida proved that there exists a unique Hida family passing through classical points with weight ≥ 2.
- Hida explicitly constructed a CM/Eisenstein Hida family.
- How does one answer the above question by simply considering the Galois representation ρ_F (which contains all the info about F)?
- If the Galois representation ρ_f, associated to a classical specialization of weight ≥ 2, induced from a Hecke character of an imaginary quadratic field, then is ρ_F also induced from a Λ-adic Hecke character?
- If a classical specialization (with weight ≥ 2) of ρ_F is reducible, is ρ_F reducible? (Yes, due to Bellaïche–Chenevier)

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- Hida proved that there exists a unique Hida family passing through classical points with weight ≥ 2.
- Hida explicitly constructed a CM/Eisenstein Hida family.
- How does one answer the above question by simply considering the Galois representation ρ_F (which contains all the info about F)?
- If the Galois representation ρ_f, associated to a classical specialization of weight ≥ 2, induced from a Hecke character of an imaginary quadratic field, then is ρ_F also induced from a Λ-adic Hecke character?
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Rigidity: do properties propagate along p-adic families?

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$$\mathrm{GSp}_4 = \left\{ g \in \mathrm{GL}_4, \text{ such that } g^T \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} g = \lambda(g) \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} \right\}.$$

If $ad - bc = a'd' - b'c'$, then

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Suppose f_0 and g_0 satisfy:

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Then, the following Galois representations have the same determinant:

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- (Laumon, Taylor, Weissauer) One can attach a Galois representation $\mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q}) \to \mathrm{GSp}_4(\overline{\mathbb{Q}}_p)$ to Siegel cuspidal eigen forms over GSp_4 with weight (κ_1, κ_2) with $\kappa_1 \geq \kappa_2 \geq 3$.
- Is there a cuspidal Siegel form whose Galois representation is isomorphic to

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  - (Roberts) Establishing Arthur's multiplicity formula for the Yoshida packet.

$$(\kappa_1, \kappa_2) = \left(\frac{\mathrm{wt}(f_0)+\mathrm{wt}(g_0)}{2}, \frac{\mathrm{wt}(f_0)-\mathrm{wt}(g_0)}{2} + 2\right).$$

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Hida theory for GSp_4 (Hida, Pilloni)

- A Siegel cusp form with weight (κ_1, κ_2) is p-ordinary if the p-adic valuations of the Hecke polynomial at p:

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- The Yoshida lift of f_0 and g_0 turns out to be p-ordinary.
- One can apply Hida theory for GSp_4 (Tilouine–Urban, Hida, Pilloni). Again, one needs to p-stabilize. However, there could be more than one Hida family passing through an ordinary form.
- $\mathbb{T}$: the $\mathbb{Z}_p[[t_1, t_2]]$-adic Hecke algebra generated by $T_{l,1}$, $T_{l,2}$ and S_l operators for good primes l along with the $U_{p,1}$ and $U_{p,2}$ operator.

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Congruence ideals associated to Yoshida lifts

$$\begin{tikzcd} \mathbb{T} \arrow[r, "\phi"] \arrow[d] & \mathbb{I} \arrow[d] \\ \mathbb{T}_\perp \arrow[r] & \star \end{tikzcd}$$

$$\frac{\mathbb{T}_\perp}{?} \cong \frac{\mathbb{I}}{(\text{cong ideal})}.$$

- We need ϕ to pass through a p-family of Yoshida lifts.
- So far, we only know that it passes through one classical Yoshida lift.
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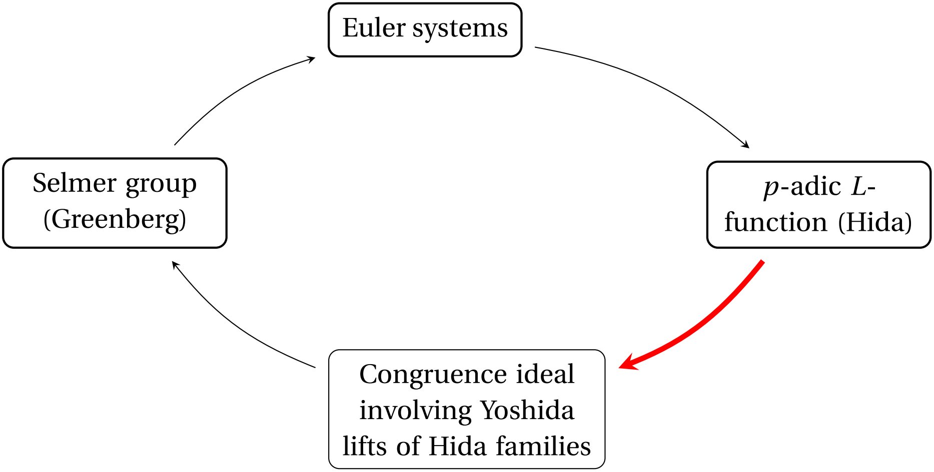
Ongoing work of Hsieh–Liu

One starting approach: a direct automorphic construction of the desired p-adic family of Siegel forms.



Question

Suppose you have a GSp_4 Hida family passing through the Yoshida lift of f_0 and g_0. Do almost all classical specializations of the Hida family correspond to Yoshida lifts?

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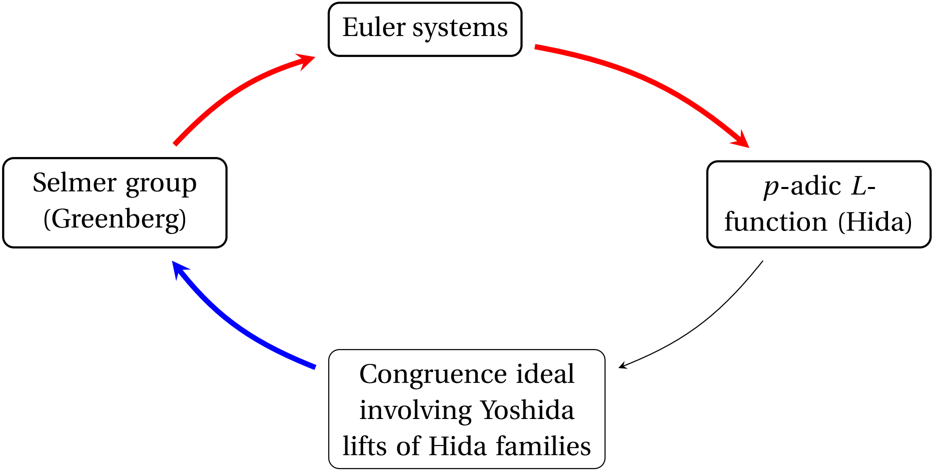

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Main results: p-adic families of Hecke eigensystems

Henceforth, we work under all the assumptions stated above.

Theorem (Hsieh–P.)

- *The p-adic family of Hecke eigensystems*

$$\phi : \mathbb{T} \to \mathbb{I}$$

  corresponds to the Yoshida lifts of F and G.
- *Every irreducible component of $\mathbb{T}_{\perp}$ does not correspond to a Yoshida lift of Hida families.*

Theorem (Hsieh–P.)

$$\mathrm{Char}\left(\mathrm{Sel}_{\mathbf{F},G}(\mathbb{Q})^{\vee}\right) \subset \text{cong ideal}^{*}. \tag{2}$$

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