Experiments with Ceresa classes of Fermat quotients

Symposium on Arithmetic Geometry and its Applications

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- C = smooth projective curve over a field $K \subset \mathbb{C}$
- g = genus of $C = \dim H^0(C, \Omega^1_C) \geq 2$
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Proof: $\iota_P^- = [-1] \circ \iota_P$ and $[-1]$ acts as $+1$ on $H^{2g-2}(J) = \wedge^{2g-2} H^1(J)$.

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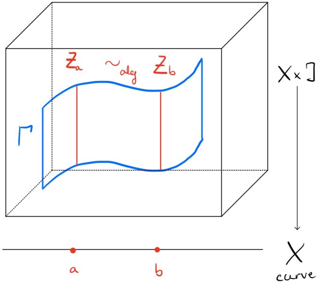

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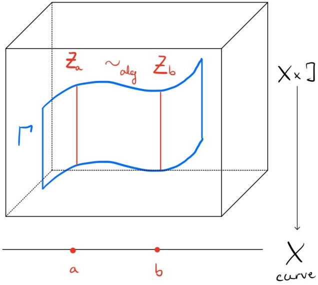

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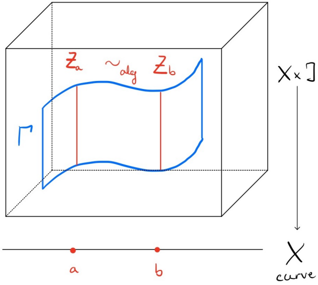

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Question: Is $\kappa(C) = 0$ in $\mathrm{Gr}_1(J)$?

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Theorem (Beauville–Schoen, 2021)

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Proof: $V = \{dx/y^2, xdx/y^2, dx/y\}$ and $\sigma(x,y) = (\zeta_9^{-3} x, \zeta_9^{-1})$. $\square$

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Let $m \in \mathbb{N}$, $0 < a, b < m$, $\gcd(m, a, b, a+b) = 1$.

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\begin{array}{llcl}
f^m_{a,b}\colon & F^m\colon X^m + Y^m + Z^m = 0 & \xrightarrow{\mu_m} & C^m_{a,b}\colon v^m = (-1)^{a+b}u^a(1-u)^b \\
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Observation: $C^m_{a,b}$ is hyperelliptic if and only if $(a,b) \sim_m (1,1)$ or $m = 2n$ and $(a,b) \sim_m (1,n)$.

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Upper bounds: $\#\,\mathrm{Fix}(\sigma_9) = 81$, $\#\,\mathrm{Fix}(\sigma_{12}) = 36$, $\#\,\mathrm{Fix}(\sigma_{15}) = 2025$.

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E.g. $L((M^9_{1,2})_{\mathbb{Q}(\zeta_9)^+}, s) = L(\tau^9_{1,2}\tau^9_{2,4}\tau^9_{5,1}/\mathbb{Q}(\zeta_9), s)$, sign $= -1$, $L'(2) \neq 0$.

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$$\kappa(C^{12}_{1,3}) = 0 \in \mathrm{Gr}_1(J^{12}_{1,3}) \otimes \mathbb{Q}$$

Proof: Adaptation of Beauville–Schoen's proof that $\kappa(C^9_{1,2})$ is torsion.
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There can be no such examples for $g = 5$ or $g \geq 21$ (Beauville, 2022).

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Thank you for your attention!

Thanks to the organizers for a wonderful week at CIRM!