Arithmetic of equations in a large number of variables

SAGA - Diego Izquierdo

CMLS - École Polytechnique

February 10th 2023

The C_i property

Definition (Artin-Lang, 1950's)

Let $i \geq 0$ be an integer.

The C_i property

Definition (Artin-Lang, 1950's)

Let $i \geq 0$ be an integer. We say that a field k has **property** C_i if,

The C_i property

Definition (Artin-Lang, 1950's)

Let $i \geq 0$ be an integer. We say that a field k has **property** C_i if, for every $n, d \geq 1$ such that $n \geq d^i$,

The C_i property

Definition (Artin-Lang, 1950's)

Let $i \geq 0$ be an integer. We say that a field k has **property** C_i if, for every $n, d \geq 1$ such that $n \geq d^i$, every hypersurface of degree d in $\mathbb{P}^n_k$ has a rational point.

The C_i property

Definition (Artin-Lang, 1950's)

Let $i \geq 0$ be an integer. We say that a field k has **property** C_i if, for every $n, d \geq 1$ such that $n \geq d^i$, every hypersurface of degree d in $\mathbb{P}^n_k$ has a rational point.

Field	C_i properties	

The C_i property

Definition (Artin-Lang, 1950's)

Let $i \geq 0$ be an integer. We say that a field k has **property** C_i if, for every $n, d \geq 1$ such that $n \geq d^i$, every hypersurface of degree d in $\mathbb{P}^n_k$ has a rational point.

Field	C_i properties	
$k = \overline{k}$	C_0	

The C_i property

Definition (Artin-Lang, 1950's)

Let $i \geq 0$ be an integer. We say that a field k has **property** C_i if, for every $n, d \geq 1$ such that $n \geq d^i$, every hypersurface of degree d in $\mathbb{P}^n_k$ has a rational point.

Field	C_i properties	
$k = \overline{k}$	C_0	
Finite fields	C_1 (Chevalley-Warning, 1935)	

The C_i property

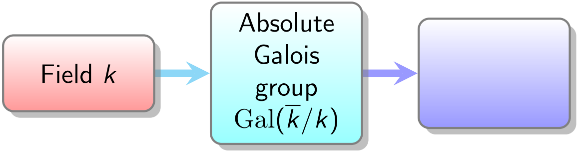
Definition (Artin-Lang, 1950's)

Let $i \geq 0$ be an integer. We say that a field k has **property** C_i if, for every $n, d \geq 1$ such that $n \geq d^i$, every hypersurface of degree d in $\mathbb{P}^n_k$ has a rational point.

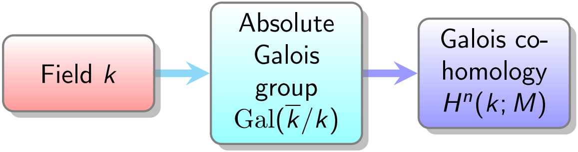
Field	C_i properties	
$k = \overline{k}$	C_0	
Finite fields	C_1 (Chevalley-Warning, 1935)	
$\mathrm{trdeg}(K/k) = \delta$	k is $C_i \Rightarrow K$ is $C_{i+\delta}$ (Tsen-Lang-Nagata, 1957)	

Cohomological dimension

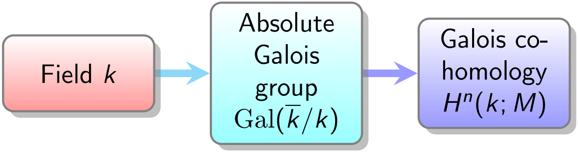
Cohomological dimension



Cohomological dimension



Cohomological dimension



Cohomological dimension of k :

$$\mathrm{cd}(k) := \max\{n \geq 0 : \exists M \text{ finite}, \; H^n(k,M) \neq 0\}.$$

The C_i property and the cohomological dimension

Definition (Artin-Lang, 1950's)

Let $i \geq 0$ be an integer. We say that a field k has **property** C_i if, for every $n, d \geq 1$ such that $n \geq d^i$, every hypersurface of degree d in $\mathbb{P}^n_k$ has a rational point.

Field	C_i properties	Coh. dim.
$k = \overline{k}$	C_0	
Finite fields	C_1 (Chevalley-Warning, 1935)	
$\mathrm{trdeg}(K/k) = \delta$	k is $C_i \Rightarrow K$ is $C_{i+\delta}$ (Tsen-Lang-Nagata, 1957)	

The C_i property and the cohomological dimension

Definition (Artin-Lang, 1950's)

Let $i \geq 0$ be an integer. We say that a field k has **property** C_i if, for every $n, d \geq 1$ such that $n \geq d^i$, every hypersurface of degree d in $\mathbb{P}^n_k$ has a rational point.

Field	C_i properties	Coh. dim.
$k = \overline{k}$	C_0	0
Finite fields	C_1 (Chevalley-Warning, 1935)	
$\mathrm{trdeg}(K/k) = \delta$	k is $C_i \Rightarrow K$ is $C_{i+\delta}$ (Tsen-Lang-Nagata, 1957)	

The C_i property and the cohomological dimension

Definition (Artin-Lang, 1950's)

Let $i \geq 0$ be an integer. We say that a field k has **property** C_i if, for every $n, d \geq 1$ such that $n \geq d^i$, every hypersurface of degree d in $\mathbb{P}^n_k$ has a rational point.

Field	C_i properties	Coh. dim.
$k = \overline{k}$	C_0	0
Finite fields	C_1 (Chevalley-Warning, 1935)	1
$\mathrm{trdeg}(K/k) = \delta$	k is $C_i \Rightarrow K$ is $C_{i+\delta}$ (Tsen-Lang-Nagata, 1957)	

The C_i property and the cohomological dimension

Definition (Artin-Lang, 1950's)

Let $i \geq 0$ be an integer. We say that a field k has **property C_i** if, for every $n, d \geq 1$ such that $n \geq d^i$, every hypersurface of degree d in $\mathbb{P}^n_k$ has a rational point.

Field	C_i properties	Coh. dim.
$k = \overline{k}$	C_0	0
Finite fields	C_1 (Chevalley-Warning, 1935)	1
$\mathrm{trdeg}(K/k) = \delta$	k is $C_i \Rightarrow K$ is $C_{i+\delta}$ (Tsen-Lang-Nagata, 1957)	$\mathrm{cd}(K) \leq \mathrm{cd}(k) + \delta$

The C_i property and the cohomological dimension

Naive question

Is it true that, for every (perfect) field k and every $i \geq 0$:

$$k \text{ is } C_i \Leftrightarrow \mathrm{cd}(k) \leq i?$$

The C_i property and the cohomological dimension

Naive question

Is it true that, for every (perfect) field k and every $i \geq 0$:

$$k \text{ is } C_i \Leftrightarrow \mathrm{cd}(k) \leq i?$$

$\Leftarrow$ **No** : p-adic fields are not C_2 (Terjanian 1966).

The C_i property and the cohomological dimension

Naive question

Is it true that, for every (perfect) field k and every $i \geq 0$:

$$k \text{ is } C_i \Leftrightarrow \mathrm{cd}(k) \leq i?$$

- $\Leftarrow$ **No** : p-adic fields are not C_2 (Terjanian 1966).
- $\Rightarrow$ **Question raised by Serre in 1960 :**

The C_i property and the cohomological dimension

Naive question

Is it true that, for every (perfect) field k and every $i \geq 0$:

$$k \text{ is } C_i \Leftrightarrow \mathrm{cd}(k) \leq i?$$

$\Leftarrow$ **No :** p-adic fields are not C_2 (Terjanian 1966).

$\Rightarrow$ **Question raised by Serre in 1960 :**

- yes for $i = 1$;

The C_i property and the cohomological dimension

Naive question

Is it true that, for every (perfect) field k and every $i \geq 0$:

$$k \text{ is } C_i \Leftrightarrow \mathrm{cd}(k) \leq i?$$

$\Leftarrow$ **No :** p-adic fields are not C_2 (Terjanian 1966).

$\Rightarrow$ **Question raised by Serre in 1960 :**

- yes for $i = 1$;
- yes for $i = 2$ (Suslin, Joukhovitski, 2006) ;

The C_i property and the cohomological dimension

Naive question

Is it true that, for every (perfect) field k and every $i \geq 0$:

$$k \text{ is } C_i \Leftrightarrow \mathrm{cd}(k) \leq i?$$

$\Leftarrow$ **No :** p-adic fields are not C_2 (Terjanian 1966).

$\Rightarrow$ **Question raised by Serre in 1960 :**

- yes for $i = 1$;
- yes for $i = 2$ (Suslin, Joukhovitski, 2006) ;
- open for $i \geq 3$.

The Kato and Kuzumaki's properties

Definition (Artin-Lang, 1950's)

Let $i \geq 0$ be an integer. We say that a field k has property C_i if, for every $n, d \geq 1$ such that $n \geq d^i$, every hypersurface X of degree d in $\mathbb{P}^n_k$ has a rational point.

The C_i property and the cohomological dimension

Definition (Artin-Lang, 1950's)

Let $i \geq 0$ be an integer. We say that a field k has property C_i if, for every $n, d \geq 1$ such that $n \geq d^i$, every hypersurface X of degree d in $\mathbb{P}^n_k$ has a rational point.

The C_i property and the cohomological dimension

Definition (Kato-Kuzumaki, 1986)

Let $i \geq 0$ be an integer. We say that a field k has property C_i^0 if, for every $n, d \geq 1$ such that $n \geq d^i$, every hypersurface X of degree d in $\mathbb{P}_k^n$ has a 0-cycle of degree 1, ie X has points in finite extensions of k of coprime degrees.

The C_i property and the cohomological dimension

Definition (Kato-Kuzumaki, 1986)

Let $i \geq 0$ be an integer. We say that a field k has property C_i^0 if, for every $n, d \geq 1$ such that $n \geq d^i$, every hypersurface X of degree d in $\mathbb{P}_k^n$ satisfies the following property :

$$\mathbb{Z} = \left\langle \mathrm{im}\left(\mathbb{Z} \xrightarrow{\times[l:k]} \mathbb{Z} \right) \middle| \begin{array}{l} l/k \text{ finite extension} \\ \text{such that } X(l) \neq \emptyset \end{array} \right\rangle.$$

The C_i property and the cohomological dimension

Definition (Kato-Kuzumaki, 1986)

Let $i \geq 0$ be an integer. We say that a field k has property C_i^1 if, for every $n, d \geq 1$ such that $n \geq d^i$, every hypersurface X of degree d in $\mathbb{P}^n_k$ satisfies the following property :

$$k^\times = \left\langle \mathrm{im}\left(l^\times \xrightarrow{N_{l/k}} k^\times \right) \middle| \begin{array}{l} l/k \text{ finite extension} \\ \text{such that } X(l) \neq \emptyset \end{array} \right\rangle .$$

The C_i property and the cohomological dimension

Definition (Kato-Kuzumaki, 1986)

Let $i \geq 0$ be an integer. We say that a field k has property C_i^1 if, for every $n, d \geq 1$ such that $n \geq d^i$, every hypersurface X of degree d in $\mathbb{P}^n_k$ satisfies the following property :

$$k^\times = \left\langle \mathrm{im}\left(l^\times \xrightarrow{N_{l/k}} k^\times \right) \middle| \begin{array}{c} l/k \text{ finite extension} \\ \text{such that } X(l) \neq \emptyset \end{array} \right\rangle.$$

$$K_q(k) := \frac{(k^\times)^{\otimes q}}{\langle x_1 \otimes ... \otimes x_q | \exists i, j, i \neq j, x_i + x_j = 1 \rangle}.$$

The C_i property and the cohomological dimension

Definition (Kato-Kuzumaki, 1986)

Let $i \geq 0$ be an integer. We say that a field k has property C_i^q if, for every $n, d \geq 1$ such that $n \geq d^i$, every hypersurface X of degree d in $\mathbb{P}_k^n$ satisfies the following property :

$$K_q(k) = \left\langle \mathrm{im}(K_q(l) \xrightarrow{N_{l/k}} K_q(k)) \,\middle|\, \begin{array}{l} l/k \text{ finite extension} \\ \text{such that } X(l) \neq \emptyset \end{array} \right\rangle.$$

$$K_q(k) := \frac{(k^\times)^{\otimes q}}{\langle x_1 \otimes \ldots \otimes x_q \mid \exists i, j, i \neq j, x_i + x_j = 1 \rangle}.$$

Kato and Kuzumaki's conjecture

Conjecture (Kato and Kuzumaki, 1986)

$$k \text{ is } C_i^q \Leftrightarrow \mathrm{cd}(k) \leq i + q.$$

Kato and Kuzumaki's conjecture

Conjecture (Kato and Kuzumaki, 1986)

$$k \text{ is } C_i^q \Leftrightarrow \mathrm{cd}(k) \leq i + q.$$

Yes for $i = 0$.

Kato and Kuzumaki's conjecture

Conjecture (Kato and Kuzumaki, 1986)

$$k \text{ is } C_i^q \Leftrightarrow \mathrm{cd}(k) \leq i + q.$$

Yes for $i = 0$.

No in general : counterexamples of :

- Merkurjev in cohomological dimension 2,
- Colliot-Thélène and Madore in cohomological dimension 1.

Questions

Questions

1. Can one prove the Kato and Kuzumaki's conjectures for some particular fields that "naturally appear" in number theory or in algebraic geometry ?

Questions

Questions

1. Can one prove the Kato and Kuzumaki's conjectures for some particular fields that "naturally appear" in number theory or in algebraic geometry ?
2. Can one further modify the C_i^q properties so that they do characterize cohomological dimension of fields ?

Questions

Question 1

Question 1 : Kato-Kuzumaki's conjecture for specific fields

p-adic fields & totally imaginary number fields :

Question 1 : Kato-Kuzumaki's conjecture for specific fields

p-adic fields & totally imaginary number fields :

They have cohomological dimension 2.

Question 1 : Kato-Kuzumaki's conjecture for specific fields

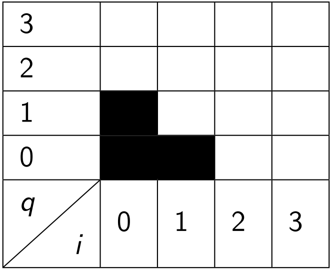
p-adic fields & totally imaginary number fields :

They have cohomological dimension 2.

3				
2				
1				
0				
q \ i	0	1	2	3

p-adic fields & totally imaginary number fields :

They have cohomological dimension 2.



Question 1 : Kato-Kuzumaki's conjecture for specific fields

p-adic fields & totally imaginary number fields :

They have cohomological dimension 2.

3	✓	✓	✓	✓
2	✓	✓	✓	✓
1	■			
0	■	■		
q \ i	0	1	2	3

✓ Bloch-Kato conjecture.

Question 1 : Kato-Kuzumaki's conjecture for specific fields

p-adic fields & totally imaginary number fields :

They have cohomological dimension 2.

q \ i	0	1	2	3
3	✓	✓	✓	✓
2	✓	✓	✓	✓
1	■	✓	✓	✓
0	■	■		

✓ Bloch-Kato conjecture.

✓ Wittenberg 2015.

Question 1 : Kato-Kuzumaki's conjecture for specific fields

p-adic fields & totally imaginary number fields :

They have cohomological dimension 2.

3	✓	✓	✓	✓
2	✓	✓	✓	✓
1		✓	✓	✓
0			?	?
q \ i	0	1	2	3

✓ Bloch-Kato conjecture.

✓ Wittenberg 2015.

Question 1 : Kato-Kuzumaki's conjecture for specific fields

Function fields :

Question 1 : Kato-Kuzumaki's conjecture for specific fields

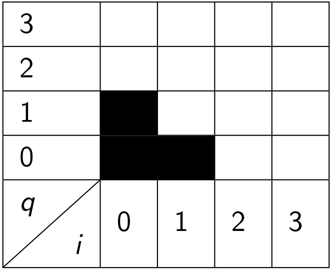
Function fields :

- $\mathbb{F}(C)$, with C curve over finite field $\mathbb{F}$.
  $\Rightarrow$ Cohomological dimension 2.

Question 1 : Kato-Kuzumaki's conjecture for specific fields

Function fields :

- $\mathbb{F}(C)$, with C curve over finite field $\mathbb{F}$.
  $\Rightarrow$ Cohomological dimension 2.



Question 1 : Kato-Kuzumaki's conjecture for specific fields

Function fields :

- $\mathbb{F}(C)$, with C curve over finite field $\mathbb{F}$.
  $\Rightarrow$ Cohomological dimension 2.

3	✓	✓	✓	✓
2	✓	✓	✓	✓
1				
0				
q \ i	0	1	2	3

✓ Bloch-Kato conjecture.

Question 1 : Kato-Kuzumaki's conjecture for specific fields

Function fields :

- $\mathbb{F}(C)$, with C curve over finite field $\mathbb{F}$.
  $\Rightarrow$ Cohomological dimension 2.

3	✓	✓	✓	✓
2	✓	✓	✓	✓
1			✓	✓
0			✓	✓
q \ i	0	1	2	3

✓ Bloch-Kato conjecture.

✓ Tsen-Lang-Nagata Theorem.

Question 1 : Kato-Kuzumaki's conjecture for specific fields

Function fields :

- $\mathbb{F}(C)$, with C curve over finite field $\mathbb{F}$.
  $\Rightarrow$ Cohomological dimension 2.

3	✓	✓	✓	✓
2	✓	✓	✓	✓
1		✓	✓	✓
0			✓	✓
q \ i	0	1	2	3

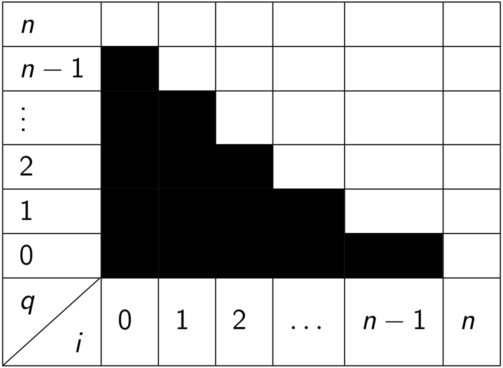
✓ Bloch-Kato conjecture.
✓ Tsen-Lang-Nagata Theorem.
✓ I. 2018.

Question 1 : Kato-Kuzumaki's conjecture for specific fields

- $\mathbb{C}(V)$, with V a complex variety of dimension n.
  $\Rightarrow$ Cohomological dimension n.

Question 1 : Kato-Kuzumaki's conjecture for specific fields

- $\mathbb{C}(V)$, with V a complex variety of dimension n.
  $\Rightarrow$ Cohomological dimension n.



Question 1 : Kato-Kuzumaki's conjecture for specific fields

- $\mathbb{C}(V)$, with V a complex variety of dimension n.
  $\Rightarrow$ Cohomological dimension n.

n	✓	✓	✓	✓	✓	✓
$n-1$	■					
$\vdots$	■	■				
2	■	■	■			
1	■	■	■	■		
0	■	■	■	■	■	
q \ i	0	1	2	$\dots$	$n-1$	n

✓ Bloch-Kato conjecture.

Question 1 : Kato-Kuzumaki's conjecture for specific fields

- $\mathbb{C}(V)$, with V a complex variety of dimension n.
  $\Rightarrow$ Cohomological dimension n.

n	✓	✓	✓	✓	✓	✓
$n-1$	■					✓
$\vdots$	■	■				✓
2	■	■	■			✓
1	■	■	■	■		✓
0	■	■	■	■	■	✓
q / i	0	1	2	$\dots$	$n-1$	n

✓ Bloch-Kato conjecture.

✓ Tsen-Lang-Nagata Theorem.

Question 1 : Kato-Kuzumaki's conjecture for specific fields

- $\mathbb{C}(V)$, with V a complex variety of dimension n.
  $\Rightarrow$ Cohomological dimension n.

n	✓	✓	✓	✓	✓	✓
$n-1$		✓	✓	✓	✓	✓
$\vdots$			✓	✓	✓	✓
2				✓	✓	✓
1					✓	✓
0						✓
q \ i	0	1	2	...	$n-1$	n

✓ Bloch-Kato conjecture.

✓ Tsen-Lang-Nagata Theorem.

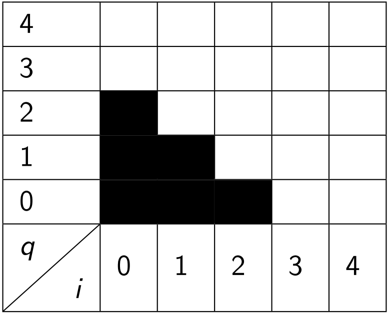
✓ I. 2018.

Question 1 : Kato-Kuzumaki's conjecture for specific fields

- $k(C)$, with C curve over p-adic field k.
  $\Rightarrow$ Cohomological dimension 3.

Question 1 : Kato-Kuzumaki's conjecture for specific fields

- $k(C)$, with C curve over p-adic field k.
  $\Rightarrow$ Cohomological dimension 3.



Question 1 : Kato-Kuzumaki's conjecture for specific fields

- $k(C)$, with C curve over p-adic field k.
  $\Rightarrow$ Cohomological dimension 3.

4	✓	✓	✓	✓	✓
3	✓	✓	✓	✓	✓
2	■				
1	■	■			
0	■	■	■		
q \ i	0	1	2	3	4

✓ Bloch-Kato conjecture.

Question 1 : Kato-Kuzumaki's conjecture for specific fields

- $k(C)$, with C curve over p-adic field k.
  $\Rightarrow$ Cohomological dimension 3.

q \ i	0	1	2	3	4
4	✓	✓	✓	✓	✓
3	✓	✓	✓	✓	✓
2	■		✓	✓	✓
1	■	■			
0	■	■	■		

✓ Bloch-Kato conjecture.

✓ I.-Lucchini Arteche 2022.

Question 1 : Kato-Kuzumaki's conjecture for specific fields

- $k(C)$, with C curve over p-adic field k.
  $\Rightarrow$ Cohomological dimension 3.

4	✓	✓	✓	✓	✓
3	✓	✓	✓	✓	✓
2		⋆	✓	✓	✓
1					
0					
q \ i	0	1	2	3	4

✓ Bloch-Kato conjecture.

✓ I.-Lucchini Arteche 2022.

⋆ I.-Lucchini Arteche 2022 : true under the assumption $C(k) \neq \emptyset$, unknown otherwise.

Question 1 : Kato-Kuzumaki's conjecture for specific fields

- $k(C)$, with C curve over p-adic field k.
  $\Rightarrow$ Cohomological dimension 3.

4	✓	✓	✓	✓	✓
3	✓	✓	✓	✓	✓
2	■	⋆	✓	✓	✓
1	■	■	?	?	?
0	■	■	■	?	?
q \ i	0	1	2	3	4

✓ Bloch-Kato conjecture.

✓ I.-Lucchini Arteche 2022.

⋆ I.-Lucchini Arteche 2022 : true under the assumption $C(k) \neq \emptyset$, unknown otherwise.

Questions

Question 2

Question 2 : Modifying the C_i^q properties

Definition

Let q be a non-negative integer. We say that k has the C_1^q-**property** if :

$$\forall n, d \geq 1 \text{ such that } n \geq d,$$
$$\forall X \subset \mathbb{P}^n_k \text{ hypersurface of degree } d,$$
$$K_q(k) = \left\langle \mathrm{im}(K_q(l) \xrightarrow{N_{l/k}} K_q(k)) \,\middle|\, \begin{array}{l} l/k \text{ finite extension} \\ \text{such that } X(l) \neq \emptyset \end{array} \right\rangle.$$

Question 2 : Modifying the C_i^q properties

Definition

Let q be a non-negative integer. We say that k has the C_1^q-**property** if :

$$\forall n, d \geq 1 \text{ such that } n \geq d,$$

$$\forall X \subset \mathbb{P}^n_k \text{ hypersurface of degree } d,$$

$$K_q(k) = \left\langle \mathrm{im}(K_q(l) \xrightarrow{N_{l/k}} K_q(k)) \;\middle|\; \begin{array}{l} l/k \text{ finite extension} \\ \text{such that } X(l) \neq \emptyset \end{array} \right\rangle.$$

Question 2 : Modifying the C_i^q properties

Definition (I., Lucchini Arteche (2020))

Let q be a non-negative integer. We say that k has the $C^q_{\mathbf{HS}}$-**property** if :

$$\sout{\forall n, d \geq 1 \text{ such that } n \geq d},$$

$$\forall X \text{ homogenous space under a connected linear } k\text{-group } G$$

$$K_q(k) = \left\langle \mathrm{im}(K_q(l) \xrightarrow{N_{l/k}} K_q(k)) \;\middle|\; \begin{array}{l} l/k \text{ finite extension} \\ \text{such that } X(l) \neq \emptyset \end{array} \right\rangle.$$

Question 2 : Modifying the C_i^q properties

Definition (I., Lucchini Arteche (2020))

Let q be a non-negative integer. We say that k has the $C^q_{\mathbf{HS}}$-**property** if :

$$\sout{\forall n, d \geq 1 \text{ such that } n \geq d},$$

$$\forall X \text{ homogenous space under a connected linear } k\text{-group } G$$

$$K_q(k) = \left\langle \operatorname{im}(K_q(l) \xrightarrow{N_{l/k}} K_q(k)) \;\middle|\; \begin{array}{l} l/k \text{ finite extension} \\ \text{such that } X(l) \neq \emptyset \end{array} \right\rangle.$$

Theorem (I., Lucchini Arteche (2020))

A perfect field has the C^q_{HS}-property if, and only if, it has cohomological dimension at most $q+1$.

Question 2 : Modifying the C_i^q properties

Remarks

(a) Imperfect fields.

Question 2 : Modifying the C_i^q properties

Remarks

(a) Imperfect fields.

(b) For $q = 0$, recover the 0-cycle version of a classical **theorem of Steinberg and Springer**.

Question 2 : Modifying the C_i^q properties

Remarks

(a) Imperfect fields.

(b) For $q = 0$, recover the 0-cycle version of a classical **theorem of Steinberg and Springer**.

(c) For $q = 1$, recover a **theorem of Suslin** (**and Gille** in positive characteristic).

Question 2 : Modifying the C_i^q properties

Remarks

(a) Imperfect fields.

(b) For $q = 0$, recover the 0-cycle version of a classical **theorem of Steinberg and Springer**.

(c) For $q = 1$, recover a **theorem of Suslin** (**and Gille** in positive characteristic).

(d) For p-adic fields and totally imaginary number fields, recover a **theorem of Wittenberg**.

Question 2 : Modifying the C_i^q properties

Remarks

(a) Imperfect fields.

(b) For $q = 0$, recover the 0-cycle version of a classical **theorem of Steinberg and Springer**.

(c) For $q = 1$, recover a **theorem of Suslin** (**and Gille** in positive characteristic).

(d) For p-adic fields and totally imaginary number fields, recover a **theorem of Wittenberg**.

(e) To replace the C_2^q property, use **principal homogeneous spaces under semi-simple simply connected groups** (preprint almost ready with Lucchini Arteche).

Thank you

Thank you for your attention !