SAGA: PROBLEM SESSION II

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Problem 1 (J. Voight)**.** Take the family

$$C_a : y^2 = x(x^4 - 1)(x^4 + 2ax^2 + 1)$$

for $a \in \mathbb{P}^1 \setminus \{\pm 1, \infty\}$. We can prove that if $p \neq 2$ then C_a has potentially good reduction. Is this still true at $p = 2$? If not, what types of reduction are possible?

Problem 2 (S. Le Fourn)**.** How sharp is Minkowski's bound? Let $K/\mathbb{Q}$ be a degree 4 extension, with signature $(0, 2)$, let $I \leq \mathcal{O}_K$ be an ideal, $0 \neq \alpha \in I$. How sharp is $\min\left\{\frac{|\alpha|}{|I|}, \alpha \in I, \alpha \neq 0\right\} \leq |D_K|^{1/2} c$? This is minimal norm of an ideal in the class that is inverse to the class of I.

Take $\tau_1, \tau_2 : K \to \mathbb{C}$ to be two (non conjugate) embeddings, and look at

$$\tau_1 \times \tau_2 : \mathcal{O}_K \to \mathbb{C} \times \mathbb{C}.$$

Then we have

$$|\alpha|_1 = |(\tau_1(\alpha), \tau_2(\alpha))|_{L^1} = |\tau_1(\alpha)| + |\tau_2(\alpha)| \geq 2 |\alpha|^{1/4}$$

How sharp is this? What can be said about the quantity

$$\operatorname*{avg}_{I \in \mathrm{Cl}_K} \min \left\{ \frac{|\alpha|_1}{|I|^{1/4}} \ : \ \alpha \in I \right\} =: c(K)$$

in families as $D_K \to \infty$?

Hint (*J. Voight*): Consider looking at shapes of number fields?

Addendum (*K. Khuri-Makdisi*): What about successive minima?

Problem 3 (A. Skorobogatov)**.** I have a surface

$$x^4 + y^4 + z^4 + w^4 = 0,$$

does it have potentially good reduction at 2? Can we say something about potentially good reduction in families of twists?

Hint (*E. Lorenzo Garcia*): Fibre into elliptic curves with CM, since $x^4 + y^4 = C$ has potentially good reduction.

Hint (*J. Voight*): Use the fact that it is (potentially) Kummer.

Problem 4 (S. Anni)**.** Assume that N is the conductor of an L-function associated to an abelian variety A/K of dimension g. Then we have a lower bound of Mestre: $N > 10.329^g$. Then what is the smallest conductor for $g = 3$ or $g = 4$?

The smallest we know are as follows:

- $g = 1$: $N = 11$ (this is smallest!);
- $g = 2$: $N = 11^2$ (this might be smallest? Reference: Farmer-Koutsoliotas); the difference is small in any case.
- $g = 3$: $1100 \leq N \leq 11^3 = 1331$;
- $g = 4$: $11355 \leq N \leq 11^4 = 14641$.

Same question for Jacobians, etc.?

Hint (*J. Voight*): The method of L-functions from nothing of Farmer-Koutsoliotas should make this algorithmically decidable. Possibly not in our lifetimes, but decidable.

Problem 5 (J. Lang)**.** Let $E/\mathbb{Q}$ be an elliptic curve with (P)CM. Let ψ be the associated Hecke character. There are two constructions for the motive attached to ψ^2. The first is cut out of $E \times E$, the second is obtained via the relative universal elliptic curve over the modular curve $\tilde{E}/X(1)$. Can we find the map from the first to the second?

(1) I have heard that this is an exercise for the right person in the case of $y^2 = x^3 \pm x$, can anyone elaborate?
(2) Can one find a nice family for which we can write down anexplicit map? For example for an infinite family of CM abelian varieties (say, of GL_2-type)?

Hint (*D. Lilienfeldt*): try Bertolini–Darmon–Prasanna for (1).

Problem 6 (D. Festi)**.** Let K be a field, $R = K[x_1, \ldots, x_n]$, (even $n = 1$), let $f_1, \ldots, f_m \in R$. Moreover, for $I \subset \{1, \ldots, m\}$, assume there is a homomorphism

$$\phi_I : \mathrm{Frac}(R) \to \mathrm{Frac}(R),$$
$$\prod_{i \in I} f_i \mapsto \square.$$

Then does there exist a K-hom $\phi : \mathrm{Frac}(R) \to \mathrm{Frac}(R)$ such that $\phi(f_i) = \square$ for all $i \in \{1, \ldots, n\}$?