SAGA - Problem Session I

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Problem 1 (Nicolas Billerey)**.** Let $k \geq 2$ be an even integer and p number. Then characterise or find examples of $f \in S_k^{\text{new}}(\Gamma_0(M))$ such that

$$a_p(f) \equiv 1 + \ell^{k-1} \mod p$$

for every $\ell \nmid Mp$. There are results known for $k = 2$:

- M is prime in the list if and only if $p \nmid \text{Numerator}\left(\frac{M-1}{12}\right)$;
- M^2 in the list, M is prime iff $M \equiv \pm 1 \mod p$ for $p \geq 5$;
- if $q \nmid Mp$ and Mq in the list then $(q^k - 1)(q^{k-2} - 1) \equiv 0 \mod p$.

Problem 2 (Samuele Anni)**.** Let K be a number field and G a non-solvable group. Give an algorithm/theorem, preferably the latter, which lists all unramified G-extensions of K effectively.

Problem 3 (Vladimir Dokchitser)**.** $E/\mathbb{Q}_p$ an elliptic curve, say given by

$$E : y^2 = x^3 + ax^2 + bx + c.$$

If we vary (p-adically) the coefficients a, b, c a little bit, then can I ensure that the special fibre is unchanged? Indeed, for elliptic curves you dig out Tate's algorithm from Silverman and the answer is yes, as long as you vary the a, b, c by a sufficiently large power of p then the algorithm is unchanged.

What if I now take:

- some hyperelliptic curve $C : y^2 = f(x)$?
- non-hyperelliptic curve?
- a smooth curve generally?

Really I'd like a reference, surely this is known?

Problem 4 (David Kohel)**.** Consider $E = X_1(11) : y^2 + y = x^3 - x$, then: $\Delta_E = -11$, $j(E) = -4096/11$. We can diagonalise

$$y^2 = x^3 - \tfrac{16}{48}x + \tfrac{8 \cdot 19}{864}.$$

Note that you often see a universal elliptic curve written in the form

$$y^2 + xy = x^3 - 36\tfrac{1}{j-123}x - \tfrac{1}{j-123}$$

$$\downarrow$$

$$y^2 = x^3 - \tfrac{j}{48(j-123)}x + \tfrac{j}{864(j-123)}.$$

Recall, $j(q) = q^{-1} + 744 + \dots$. On my example, I can solve for

$$t_E = \tfrac{1}{j_E} = \tfrac{-11}{4096} = q - 744 + \dots /\mathbb{Z}_{11}.$$

Tate describes a curve

$$y^2 = x^3 - \frac{E_4(q)}{48}x + \frac{E_6(q)}{864}.$$

Take $\Delta(T) = \Delta(q) = q + \dots$. Get $\Delta(E_j) = \Delta(q)\left(\frac{E_4(q)}{E_6(q)}\right)^6$ so this is a twist of E_j.

What is this twisting parameter p-adically? If $j_E \in \mathbb{Q}$, then is $\lambda(q_E) = \frac{E_4(q_E)}{E_6(q_E)}$ transcendental?