When are the natural embeddings of classical invariant rings pure?

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Consider a reductive linear algebraic group G acting linearly on a polynomial ring S over an infinite field; key examples are the general linear group, the symplectic group, the orthogonal group, and the special linear group, with the classical representations as in Weyl's book: for the general linear group, consider a direct sum of copies of the standard representation and copies of the dual; in the other cases take copies of the standard representation. The invariant rings in the respective cases are determinantal rings, rings defined by Pfaffians of alternating matrices, symmetric determinantal rings, and the Plücker coordinate rings of Grassmannians; these are the classical invariant rings of the title, with $S^G \subseteq S$ being the natural embedding.

Over a field of characteristic zero, a reductive group is linearly reductive, and it follows that the invariant ring S^G is a pure subring of S, equivalently, S^G is a direct summand of S as an S^G-module. Over fields of positive characteristic, reductive groups are typically no longer linearly reductive. We determine, in the positive characteristic case, precisely when the inclusion of S^G in S is pure. This is joint work with Melvin Hochster, Jack Jeffries, and Vaibhav Pandey.