The Analytic Spread of a Filtration

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Let (R, m_R) be a Noetherian local ring. We extend the analytic spread $\ell(I)$ of an ideal I in R to the analytic spread $\ell(\mathcal{I})$ of a (possibly non-Noetherian) filtration $\mathcal{I} = \{I_n\}$ in R. The Rees algebra of $\mathcal{I}$ is $R[\mathcal{I}] = \oplus_{n\geq 0} I_n$ and the analytic spread is $\ell(\mathcal{I}) = \dim R[\mathcal{I}]/m_R \dim R[\mathcal{I}]$.

The analytic spread of an ideal I satisfies the inequalities $\mathrm{ht}(I) \leq \ell(I) \leq \dim R$. For a general filtration, we have that $\ell(\mathcal{I}) \leq \dim R$. While the height $\mathrm{ht}(\mathcal{I})$ of a filtration is defined (it is the common height of I_n for $n \geq 1$) we can have that $\ell(\mathcal{I}) < \mathrm{ht}(\mathcal{I})$. We give examples of symbolic algebras of space curves such that all possible analytic spreads 0,1 and 2 do occur.

We show that a classical theorem of McAdam for analytic spread of ideals generalizes to **Q**-divisorial filtrations. Let R be an excellent local domain which is either of equicharacteristic zero or of dimension ≤ 3. Let $\mathcal{I} = \{I_n\}$ be a **Q**-divisorial filtration; that is, $I_n = I(\nu_1)_{a_1 n} \cap \cdots \cap I(\nu_r)_{a_r n}$ where $\nu_1, \ldots, \nu_r$ are divisorial valuations which are nonnegative on R, $a_1, \ldots, a_r$ are positive rational numbers and $I(\nu_i)_{a_i n} = \{f \in R \mid \nu_i(f) \geq a_i n\}$. Then the analytic spread $\ell(\mathcal{I}) = \dim R$ if and only if $m_R \in \mathrm{Ass}(R/I_m)$ for some positive integer m. This theorem is not true for more general filtrations. It fails even for **R**-divisorial filtrations (the a_i are positive real numbers).

Much of this work is joint with Parangama Sarkar.