On the Rajchman property for self-similar measures on $\mathbb{R}^d$

Ariel Rapaport

Technion – Israel Institute of Technology

Rajchman measures

Rajchman measures

Given a Borel probability measure ν on $\mathbb{R}^d$ its Fourier transform is denoted by $\widehat{\nu}$.

Rajchman measures

Given a Borel probability measure ν on $\mathbb{R}^d$ its Fourier transform is denoted by $\widehat{\nu}$. That is,

$$\widehat{\nu}(\xi) := \int e^{i\langle \xi, x\rangle}\, d\nu(x) \text{ for } \xi \in \mathbb{R}^d.$$

Rajchman measures

Given a Borel probability measure ν on $\mathbb{R}^d$ its Fourier transform is denoted by $\widehat{\nu}$. That is,

$$\widehat{\nu}(\xi) := \int e^{i\langle \xi, x\rangle}\, d\nu(x) \text{ for } \xi \in \mathbb{R}^d.$$

It is said that ν is a Rajchman measure if $|\widehat{\nu}(\xi)| \to 0$ as $|\xi| \to \infty$.

Given a Borel probability measure ν on $\mathbb{R}^d$ its Fourier transform is denoted by $\widehat{\nu}$. That is,

$$\widehat{\nu}(\xi) := \int e^{i\langle \xi, x\rangle}\, d\nu(x) \text{ for } \xi \in \mathbb{R}^d.$$

It is said that ν is a Rajchman measure if $|\widehat{\nu}(\xi)| \to 0$ as $|\xi| \to \infty$.

The Riemann–Lebesgue lemma says that ν is Rajchman whenever it is absolutely continuous.

Rajchman measures

Given a Borel probability measure ν on $\mathbb{R}^d$ its Fourier transform is denoted by $\widehat{\nu}$. That is,

$$\widehat{\nu}(\xi) := \int e^{i\langle \xi, x\rangle}\, d\nu(x) \text{ for } \xi \in \mathbb{R}^d.$$

It is said that ν is a Rajchman measure if $|\widehat{\nu}(\xi)| \to 0$ as $|\xi| \to \infty$.

The Riemann–Lebesgue lemma says that ν is Rajchman whenever it is absolutely continuous. For singular measures determining which ones are Rajchman is a subtle question with a long history.

Rajchman measures

Given a Borel probability measure ν on $\mathbb{R}^d$ its Fourier transform is denoted by $\widehat{\nu}$. That is,

$$\widehat{\nu}(\xi) := \int e^{i\langle \xi, x\rangle}\, d\nu(x) \text{ for } \xi \in \mathbb{R}^d.$$

It is said that ν is a Rajchman measure if $|\widehat{\nu}(\xi)| \to 0$ as $|\xi| \to \infty$.

The Riemann–Lebesgue lemma says that ν is Rajchman whenever it is absolutely continuous. For singular measures determining which ones are Rajchman is a subtle question with a long history.

Here we discuss the Rajchman property in the context of self-similar measures on $\mathbb{R}^d$.

Self-similar sets and measures

Self-similar sets and measures

A contracting similarity is a map $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ of the form $\varphi(x) = rUx + a$, where $0 < r < 1$, $U \in O(d)$ and $a \in \mathbb{R}^d$.

Self-similar sets and measures

A contracting similarity is a map $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ of the form $\varphi(x) = rUx + a$, where $0 < r < 1$, $U \in O(d)$ and $a \in \mathbb{R}^d$.

Fix a finite collection $\Phi = \{\varphi_i\}_{i=1}^{\ell}$ of contracting similarities of $\mathbb{R}^d$.

Self-similar sets and measures

A contracting similarity is a map $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ of the form $\varphi(x) = rUx + a$, where $0 < r < 1$, $U \in O(d)$ and $a \in \mathbb{R}^d$.

Fix a finite collection $\Phi = \{\varphi_i\}_{i=1}^{\ell}$ of contracting similarities of $\mathbb{R}^d$. Such a collection is called a self-similar IFS.

Self-similar sets and measures

A contracting similarity is a map $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ of the form $\varphi(x) = rUx + a$, where $0 < r < 1$, $U \in O(d)$ and $a \in \mathbb{R}^d$.

Fix a finite collection $\Phi = \{\varphi_i\}_{i=1}^{\ell}$ of contracting similarities of $\mathbb{R}^d$. Such a collection is called a self-similar IFS.

There exists a unique nonempty compact $K_\Phi \subset \mathbb{R}^d$ with

$$K_\Phi = \cup_{i=1}^{\ell} \varphi_i(K_\Phi) \, .$$

Self-similar sets and measures

A contracting similarity is a map $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ of the form $\varphi(x) = rUx + a$, where $0 < r < 1$, $U \in O(d)$ and $a \in \mathbb{R}^d$.

Fix a finite collection $\Phi = \{\varphi_i\}_{i=1}^{\ell}$ of contracting similarities of $\mathbb{R}^d$. Such a collection is called a self-similar IFS.

There exists a unique nonempty compact $K_\Phi \subset \mathbb{R}^d$ with

$$K_\Phi = \cup_{i=1}^{\ell} \varphi_i(K_\Phi) \, .$$

It is called the self-similar set or attractor corresponding to Φ.

Given a probability vector $p = (p_i)_{i=1}^{\ell}$ there exists a unique Borel probability measure μ_p on $\mathbb{R}^d$ with,

$$\mu_p = \sum_{i=1}^{\ell} p_i \cdot \varphi_i \mu_p.$$

Given a probability vector $p = (p_i)_{i=1}^{\ell}$ there exists a unique Borel probability measure μ_p on $\mathbb{R}^d$ with,

$$\mu_p = \sum_{i=1}^{\ell} p_i \cdot \varphi_i \mu_p.$$

It is called the self-similar measure corresponding to Φ and p.

Given a probability vector $p = (p_i)_{i=1}^{\ell}$ there exists a unique Borel probability measure μ_p on $\mathbb{R}^d$ with,

$$\mu_p = \sum_{i=1}^{\ell} p_i \cdot \varphi_i \mu_p.$$

It is called the self-similar measure corresponding to Φ and p.

It always holds that $\operatorname{supp}(\mu_p) \subset K_\Phi$, with equality whenever $p > 0$.

Given a probability vector $p = (p_i)_{i=1}^{\ell}$ there exists a unique Borel probability measure μ_p on $\mathbb{R}^d$ with,

$$\mu_p = \sum_{i=1}^{\ell} p_i \cdot \varphi_i \mu_p.$$

It is called the self-similar measure corresponding to Φ and p.

It always holds that $\mathrm{supp}(\mu_p) \subset K_\Phi$, with equality whenever $p > 0$.

Set $\Delta_\ell := \{(p_i)_{i=1}^{\ell} \in (0,1]^\ell \ : \ p_1 + ... p_\ell = 1\}$.

Given a probability vector $p = (p_i)_{i=1}^{\ell}$ there exists a unique Borel probability measure μ_p on $\mathbb{R}^d$ with,

$$\mu_p = \sum_{i=1}^{\ell} p_i \cdot \varphi_i \mu_p.$$

It is called the self-similar measure corresponding to Φ and p.

It always holds that $\mathrm{supp}(\mu_p) \subset K_\Phi$, with equality whenever $p > 0$.

Set $\Delta_\ell := \{(p_i)_{i=1}^{\ell} \in (0,1]^\ell \,:\, p_1 + ... p_\ell = 1\}$.

Our goal is to find necessary and sufficient algebraic conditions on Φ, under which μ_p will be Rajchman for all $p \in \Delta_\ell$.

Affine irreducibility

Affine irreducibility

Definition

We say that Φ *is* *affinely irreducible* *if there does not exist a proper affine subspace* $\mathbb{V}$ *of* $\mathbb{R}^d$ *so that* $\varphi_i(\mathbb{V}) = \mathbb{V}$ *for all* $1 \leq i \leq \ell$.

Affine irreducibility

Definition

We say that Φ is affinely irreducible if there does not exist a proper affine subspace $\mathbb{V}$ of $\mathbb{R}^d$ so that $\varphi_i(\mathbb{V}) = \mathbb{V}$ for all $1 \le i \le \ell$.

Easy to see that Φ is not affinely irreducible if and only if its attractor K_Φ is contained in a proper affine subspace $\mathbb{V}$ of $\mathbb{R}^d$.

Affine irreducibility

Definition

We say that Φ is affinely irreducible if there does not exist a proper affine subspace $\mathbb{V}$ of $\mathbb{R}^d$ so that $\varphi_i(\mathbb{V}) = \mathbb{V}$ for all $1 \le i \le \ell$.

Easy to see that Φ is not affinely irreducible if and only if its attractor K_Φ is contained in a proper affine subspace $\mathbb{V}$ of $\mathbb{R}^d$.

If Φ is not affinely irreducible, there exists a proper linear subspace $\mathbb{V} \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$ so that $K_\Phi \subset x + \mathbb{V}$.

Affine irreducibility

Definition

We say that Φ is affinely irreducible if there does not exist a proper affine subspace $\mathbb{V}$ of $\mathbb{R}^d$ so that $\varphi_i(\mathbb{V}) = \mathbb{V}$ for all $1 \leq i \leq \ell$.

Easy to see that Φ is not affinely irreducible if and only if its attractor K_Φ is contained in a proper affine subspace $\mathbb{V}$ of $\mathbb{R}^d$.

If Φ is not affinely irreducible, there exists a proper linear subspace $\mathbb{V} \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$ so that $K_\Phi \subset x + \mathbb{V}$. Consequently, given $p \in \Delta_\ell$ we have $\mu_p(x + \mathbb{V}) = 1$,

Affine irreducibility

Definition

We say that Φ is affinely irreducible if there does not exist a proper affine subspace $\mathbb{V}$ of $\mathbb{R}^d$ so that $\varphi_i(\mathbb{V}) = \mathbb{V}$ for all $1 \le i \le \ell$.

Easy to see that Φ is not affinely irreducible if and only if its attractor K_Φ is contained in a proper affine subspace $\mathbb{V}$ of $\mathbb{R}^d$.

If Φ is not affinely irreducible, there exists a proper linear subspace $\mathbb{V} \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$ so that $K_\Phi \subset x + \mathbb{V}$. Consequently, given $p \in \Delta_\ell$ we have $\mu_p(x + \mathbb{V}) = 1$, which implies $|\widehat{\mu_p}(\xi)| = 1$ for all for all $\xi \in \mathbb{V}^\perp$.

Affine irreducibility

Definition

We say that Φ is affinely irreducible if there does not exist a proper affine subspace $\mathbb{V}$ of $\mathbb{R}^d$ so that $\varphi_i(\mathbb{V}) = \mathbb{V}$ for all $1 \leq i \leq \ell$.

Easy to see that Φ is not affinely irreducible if and only if its attractor K_Φ is contained in a proper affine subspace $\mathbb{V}$ of $\mathbb{R}^d$.

If Φ is not affinely irreducible, there exists a proper linear subspace $\mathbb{V} \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$ so that $K_\Phi \subset x + \mathbb{V}$. Consequently, given $p \in \Delta_\ell$ we have $\mu_p(x + \mathbb{V}) = 1$, which implies $|\widehat{\mu_p}(\xi)| = 1$ for all for all $\xi \in \mathbb{V}^\perp$.

Hence we always assume that Φ is affinely irreducible.

We turn to describe some relevant previous results.

We turn to describe some relevant previous results. We start with the most basic case of Bernoulli convolutions.

Bernoulli convolutions

We turn to describe some relevant previous results. We start with the most basic case of Bernoulli convolutions.

Given $\lambda \in (0,1)$, write ν_λ for the distribution of the random sum $\sum_{n\geq 0} \pm\lambda^n$, where the $\pm$ are independent unbiased random variables.

Bernoulli convolutions

We turn to describe some relevant previous results. We start with the most basic case of Bernoulli convolutions.

Given $\lambda \in (0,1)$, write ν_λ for the distribution of the random sum $\sum_{n\geq 0} \pm\lambda^n$, where the $\pm$ are independent unbiased random variables. It is called the Bernoulli convolution with parameter λ.

Bernoulli convolutions

We turn to describe some relevant previous results. We start with the most basic case of Bernoulli convolutions.

Given $\lambda \in (0,1)$, write ν_λ for the distribution of the random sum $\sum_{n\geq 0} \pm\lambda^n$, where the $\pm$ are independent unbiased random variables. It is called the Bernoulli convolution with parameter λ.

ν_λ is the self-similar measure corresponding to the IFS $\{t \to \lambda t \pm 1\}$ on $\mathbb{R}$ and the probability vector $(\frac{1}{2}, \frac{1}{2})$.

Bernoulli convolutions

We turn to describe some relevant previous results. We start with the most basic case of Bernoulli convolutions.

Given $\lambda \in (0,1)$, write ν_λ for the distribution of the random sum $\sum_{n\geq 0} \pm\lambda^n$, where the $\pm$ are independent unbiased random variables. It is called the Bernoulli convolution with parameter λ.

ν_λ is the self-similar measure corresponding to the IFS $\{t \to \lambda t \pm 1\}$ on $\mathbb{R}$ and the probability vector $(\frac{1}{2}, \frac{1}{2})$.

P. Erdős (1939) proved that ν_λ is not Rajchman whenever λ^{-1} is a Pisot number different from 2.

Bernoulli convolutions

We turn to describe some relevant previous results. We start with the most basic case of Bernoulli convolutions.

Given $\lambda \in (0,1)$, write ν_λ for the distribution of the random sum $\sum_{n\geq 0} \pm\lambda^n$, where the $\pm$ are independent unbiased random variables. It is called the Bernoulli convolution with parameter λ.

ν_λ is the self-similar measure corresponding to the IFS $\{t \to \lambda t \pm 1\}$ on $\mathbb{R}$ and the probability vector $(\frac{1}{2}, \frac{1}{2})$.

P. Erdős (1939) proved that ν_λ is not Rajchman whenever λ^{-1} is a Pisot number different from 2.

Note that $\nu_{1/2}$ is absolutely continuous and in particular Rajchman.

Recall that a Pisot number, also called a Pisot–Vijayaraghavan number or a P.V. number, is an algebraic integer greater than one whose algebraic conjugates are all less than one in modulus.

Recall that a Pisot number, also called a Pisot–Vijayaraghavan number or a P.V. number, is an algebraic integer greater than one whose algebraic conjugates are all less than one in modulus.

Later R. Salem (1943) showed that if λ^{-1} is not a Pisot number then ν_λ is Rajchman,

Recall that a Pisot number, also called a Pisot–Vijayaraghavan number or a P.V. number, is an algebraic integer greater than one whose algebraic conjugates are all less than one in modulus.

Later R. Salem (1943) showed that if λ^{-1} is not a Pisot number then ν_λ is Rajchman, thus providing a characterization of the Rajchman Bernoulli convolutions.

Self-similar measures on the line

Self-similar measures on the line

Suppose that $\Phi = \{\varphi_i(t) = r_i t + a_i\}_{i=1}^{\ell}$ is a self-similar IFS on $\mathbb{R}$, with $r_i > 0$ for $1 \leq i \leq \ell$.

Self-similar measures on the line

Suppose that $\Phi = \{\varphi_i(t) = r_i t + a_i\}_{i=1}^{\ell}$ is a self-similar IFS on $\mathbb{R}$, with $r_i > 0$ for $1 \leq i \leq \ell$. Note that Φ is affinely irreducible if and only if the maps in Φ do not all have the same fixed point.

Self-similar measures on the line

Suppose that $\Phi = \{\varphi_i(t) = r_i t + a_i\}_{i=1}^{\ell}$ is a self-similar IFS on $\mathbb{R}$, with $r_i > 0$ for $1 \le i \le \ell$. Note that Φ is affinely irreducible if and only if the maps in Φ do not all have the same fixed point.

Let $\mathbf{H} \subset \mathbb{R}_{>0}$ be the group generated by the contractions $\{r_i\}_{i=1}^{\ell}$.

Suppose that $\Phi = \{\varphi_i(t) = r_i t + a_i\}_{i=1}^{\ell}$ is a self-similar IFS on $\mathbb{R}$, with $r_i > 0$ for $1 \le i \le \ell$. Note that Φ is affinely irreducible if and only if the maps in Φ do not all have the same fixed point.

Let $\mathbf{H} \subset \mathbb{R}_{>0}$ be the group generated by the contractions $\{r_i\}_{i=1}^{\ell}$. The following theorem reduces the problem of characterizing the non-Rajchman self-similar measures on $\mathbb{R}$ to the case in which $\mathbf{H}$ is cyclic.

Self-similar measures on the line

Suppose that $\Phi = \{\varphi_i(t) = r_i t + a_i\}_{i=1}^{\ell}$ is a self-similar IFS on $\mathbb{R}$, with $r_i > 0$ for $1 \leq i \leq \ell$. Note that Φ is affinely irreducible if and only if the maps in Φ do not all have the same fixed point.

Let $\mathbf{H} \subset \mathbb{R}_{>0}$ be the group generated by the contractions $\{r_i\}_{i=1}^{\ell}$. The following theorem reduces the problem of characterizing the non-Rajchman self-similar measures on $\mathbb{R}$ to the case in which $\mathbf{H}$ is cyclic.

Theorem (Jialun Li & Tuomas Sahlsten, 2019)

Suppose that Φ is affinely irreducible and that $\mathbf{H}$ is not cyclic. Then μ_p is Rajchman for every $p \in \Delta_\ell$.

Self-similar measures on the line

Suppose that $\Phi = \{\varphi_i(t) = r_i t + a_i\}_{i=1}^{\ell}$ is a self-similar IFS on $\mathbb{R}$, with $r_i > 0$ for $1 \le i \le \ell$. Note that Φ is affinely irreducible if and only if the maps in Φ do not all have the same fixed point.

Let $\mathbf{H} \subset \mathbb{R}_{>0}$ be the group generated by the contractions $\{r_i\}_{i=1}^{\ell}$. The following theorem reduces the problem of characterizing the non-Rajchman self-similar measures on $\mathbb{R}$ to the case in which $\mathbf{H}$ is cyclic.

Theorem (Jialun Li & Tuomas Sahlsten, 2019)

Suppose that Φ is affinely irreducible and that $\mathbf{H}$ is not cyclic. Then μ_p is Rajchman for every $p \in \Delta_\ell$.

The proof is based on the classical renewal theorem for transient random walks on $\mathbb{R}$.

Self-similar measures on the line

Suppose that $\Phi = \{\varphi_i(t) = r_i t + a_i\}_{i=1}^{\ell}$ is a self-similar IFS on $\mathbb{R}$, with $r_i > 0$ for $1 \leq i \leq \ell$. Note that Φ is affinely irreducible if and only if the maps in Φ do not all have the same fixed point.

Let $\mathbf{H} \subset \mathbb{R}_{>0}$ be the group generated by the contractions $\{r_i\}_{i=1}^{\ell}$. The following theorem reduces the problem of characterizing the non-Rajchman self-similar measures on $\mathbb{R}$ to the case in which $\mathbf{H}$ is cyclic.

Theorem (Jialun Li & Tuomas Sahlsten, 2019)

Suppose that Φ is affinely irreducible and that $\mathbf{H}$ is not cyclic. Then μ_p is Rajchman for every $p \in \Delta_\ell$.

The proof is based on the classical renewal theorem for transient random walks on $\mathbb{R}$. This approach was initiated by Li, who considered the Rajchman property in the context of Furstenberg measures on $\mathbb{RP}^1$.

The situation in which $\mathbf{H}$ is cyclic is considered in the following theorem.

The situation in which $\mathbf{H}$ is cyclic is considered in the following theorem.

Theorem (Julien Brémont, 2020)

Suppose that Φ is affinely irreducible and that $\mathbf{H}$ is cyclic.

The situation in which $\mathbf{H}$ is cyclic is considered in the following theorem.

Theorem (Julien Brémont, 2020)

Suppose that Φ is affinely irreducible and that $\mathbf{H}$ is cyclic. Let $r \in (0,1)$ be with $\mathbf{H} = \{r^n\}_{n\in\mathbb{Z}}$.

The situation in which $\mathbf{H}$ is cyclic is considered in the following theorem.

Theorem (Julien Brémont, 2020)

Suppose that Φ is affinely irreducible and that $\mathbf{H}$ is cyclic. Let $r \in (0,1)$ be with $\mathbf{H} = \{r^n\}_{n \in \mathbb{Z}}$. Then there exists $p \in \Delta_\ell$ so that μ_p is non-Rajchman if and only if, r^{-1} is a Pisot number

The situation in which $\mathbf{H}$ is cyclic is considered in the following theorem.

Theorem (Julien Brémont, 2020)

Suppose that Φ is affinely irreducible and that $\mathbf{H}$ is cyclic. Let $r \in (0,1)$ be with $\mathbf{H} = \{r^n\}_{n\in\mathbb{Z}}$. Then there exists $p \in \Delta_\ell$ so that μ_p is non-Rajchman if and only if, r^{-1} is a Pisot number and Φ can be conjugated by a suitable similarity to a form such that $a_i \in \mathbb{Q}(r)$ for $1 \le i \le \ell$.

The situation in which $\mathbf{H}$ is cyclic is considered in the following theorem.

Theorem (Julien Brémont, 2020)

Suppose that Φ is affinely irreducible and that $\mathbf{H}$ is cyclic. Let $r \in (0,1)$ be with $\mathbf{H} = \{r^n\}_{n\in\mathbb{Z}}$. Then there exists $p \in \Delta_\ell$ so that μ_p is non-Rajchman if and only if, r^{-1} is a Pisot number and Φ can be conjugated by a suitable similarity to a form such that $a_i \in \mathbb{Q}(r)$ for $1 \le i \le \ell$.

The last two theorems provide a complete algebraic characterization of orientation preserving self-similar IFSs on $\mathbb{R}$ for which there exists a non-Rajchman self-similar measure.

The situation in which $\mathbf{H}$ is cyclic is considered in the following theorem.

Theorem (Julien Brémont, 2020)

Suppose that Φ is affinely irreducible and that $\mathbf{H}$ is cyclic. Let $r \in (0,1)$ be with $\mathbf{H} = \{r^n\}_{n\in\mathbb{Z}}$. Then there exists $p \in \Delta_\ell$ so that μ_p is non-Rajchman if and only if, r^{-1} is a Pisot number and Φ can be conjugated by a suitable similarity to a form such that $a_i \in \mathbb{Q}(r)$ for $1 \leq i \leq \ell$.

The last two theorems provide a complete algebraic characterization of orientation preserving self-similar IFSs on $\mathbb{R}$ for which there exists a non-Rajchman self-similar measure.

It is desirable to extend this characterization to general self-similar IFSs on $\mathbb{R}^d$.

P.V. tuples

P.V. tuples

Definition

Given $k \geq 1$*, a finite collection* $\{\theta_1, ..., \theta_k\} \subset \mathbb{C}$ *of distinct algebraic integers is said to be a P.V.* k*-tuple if the following conditions are satisfied:*

P.V. tuples

Definition

Given $k \geq 1$*, a finite collection* $\{\theta_1, ..., \theta_k\} \subset \mathbb{C}$ *of distinct algebraic integers is said to be a P.V.* k*-tuple if the following conditions are satisfied:*

1. $|\theta_j| > 1$ *for* $1 \leq j \leq k$*;*

P.V. tuples

Definition

Given $k \geq 1$, *a finite collection* $\{\theta_1, ..., \theta_k\} \subset \mathbb{C}$ *of distinct algebraic integers is said to be a P.V.* k*-tuple if the following conditions are satisfied:*

1. $|\theta_j| > 1$ *for* $1 \leq j \leq k$;
2. *there exists a monic polynomial* $P \in \mathbb{Z}[X]$ *so that* $P(\theta_j) = 0$ *for* $1 \leq j \leq k$, *and* $|z| < 1$ *for* $z \in \mathbb{C} \setminus \{\theta_1, ..., \theta_k\}$ *with* $P(z) = 0$.

P.V. tuples

Definition

Given $k \geq 1$, a finite collection $\{\theta_1, ..., \theta_k\} \subset \mathbb{C}$ of distinct algebraic integers is said to be a P.V. k-tuple if the following conditions are satisfied:

1. $|\theta_j| > 1$ *for* $1 \leq j \leq k$;
2. *there exists a monic polynomial* $P \in \mathbb{Z}[X]$ *so that* $P(\theta_j) = 0$ *for* $1 \leq j \leq k$, *and* $|z| < 1$ *for* $z \in \mathbb{C} \setminus \{\theta_1, ..., \theta_k\}$ *with* $P(z) = 0$.

Note that a positive real number θ is a Pisot number precisely when $\{\theta\}$ is a P.V. 1-tuple.

Linearly invariant subspaces

As noted above, we shall always assume that our function systems are affinely irreducible.

Linearly invariant subspaces

As noted above, we shall always assume that our function systems are affinely irreducible. On the other hand, the situation in which there exists a nontrivial linearly invariant subspace should be taken into account.

Linearly invariant subspaces

As noted above, we shall always assume that our function systems are affinely irreducible. On the other hand, the situation in which there exists a nontrivial linearly invariant subspace should be taken into account.

Let $\Phi = \{\varphi_i(x) = r_i U x + a_i\}_{i=1}^{\ell}$ be a self-similar IFS on $\mathbb{R}^d$,

Linearly invariant subspaces

As noted above, we shall always assume that our function systems are affinely irreducible. On the other hand, the situation in which there exists a nontrivial linearly invariant subspace should be taken into account.

Let $\Phi = \{\varphi_i(x) = r_i U x + a_i\}_{i=1}^{\ell}$ be a self-similar IFS on $\mathbb{R}^d$, and let $\mathbb{V}$ be a linear subspace of $\mathbb{R}^d$ with $d' := \dim \mathbb{V} > 0$.

Linearly invariant subspaces

As noted above, we shall always assume that our function systems are affinely irreducible. On the other hand, the situation in which there exists a nontrivial linearly invariant subspace should be taken into account.

Let $\Phi = \{\varphi_i(x) = r_i U x + a_i\}_{i=1}^{\ell}$ be a self-similar IFS on $\mathbb{R}^d$, and let $\mathbb{V}$ be a linear subspace of $\mathbb{R}^d$ with $d' := \dim \mathbb{V} > 0$. We say that $\mathbb{V}$ is linearly invariant with respect to Φ if $U_i(\mathbb{V}) = \mathbb{V}$ for $1 \leq i \leq \ell$.

Linearly invariant subspaces

As noted above, we shall always assume that our function systems are affinely irreducible. On the other hand, the situation in which there exists a nontrivial linearly invariant subspace should be taken into account.

Let $\Phi = \{\varphi_i(x) = r_i U x + a_i\}_{i=1}^{\ell}$ be a self-similar IFS on $\mathbb{R}^d$, and let $\mathbb{V}$ be a linear subspace of $\mathbb{R}^d$ with $d' := \dim \mathbb{V} > 0$. We say that $\mathbb{V}$ is linearly invariant with respect to Φ if $U_i(\mathbb{V}) = \mathbb{V}$ for $1 \leq i \leq \ell$.

Assuming this and if $S : \mathbb{V} \to \mathbb{R}^{d'}$ is an isometry, for every $1 \leq i \leq \ell$ there exists $U'_i \in O(d')$ and $a'_i \in \mathbb{R}^{d'}$ so that

$$S \circ \pi_{\mathbb{V}} \circ \varphi_i \circ S^{-1}(x) = r_i U'_i x + a'_i \text{ for } x \in \mathbb{R}^{d'} .$$

Linearly invariant subspaces

As noted above, we shall always assume that our function systems are affinely irreducible. On the other hand, the situation in which there exists a nontrivial linearly invariant subspace should be taken into account.

Let $\Phi = \{\varphi_i(x) = r_i U x + a_i\}_{i=1}^{\ell}$ be a self-similar IFS on $\mathbb{R}^d$, and let $\mathbb{V}$ be a linear subspace of $\mathbb{R}^d$ with $d' := \dim \mathbb{V} > 0$. We say that $\mathbb{V}$ is linearly invariant with respect to Φ if $U_i(\mathbb{V}) = \mathbb{V}$ for $1 \leq i \leq \ell$.

Assuming this and if $S : \mathbb{V} \to \mathbb{R}^{d'}$ is an isometry, for every $1 \leq i \leq \ell$ there exists $U_i' \in O(d')$ and $a_i' \in \mathbb{R}^{d'}$ so that

$$S \circ \pi_{\mathbb{V}} \circ \varphi_i \circ S^{-1}(x) = r_i U_i' x + a_i' \text{ for } x \in \mathbb{R}^{d'} .$$

In particular, $\{S \circ \pi_{\mathbb{V}} \circ \varphi_i \circ S^{-1}\}_{i=1}^{\ell}$ is a self-similar IFS on $\mathbb{R}^{d'}$.

Linearly invariant subspaces

As noted above, we shall always assume that our function systems are affinely irreducible. On the other hand, the situation in which there exists a nontrivial linearly invariant subspace should be taken into account.

Let $\Phi = \{\varphi_i(x) = r_i U x + a_i\}_{i=1}^{\ell}$ be a self-similar IFS on $\mathbb{R}^d$, and let $\mathbb{V}$ be a linear subspace of $\mathbb{R}^d$ with $d' := \dim \mathbb{V} > 0$. We say that $\mathbb{V}$ is linearly invariant with respect to Φ if $U_i(\mathbb{V}) = \mathbb{V}$ for $1 \le i \le \ell$.

Assuming this and if $S : \mathbb{V} \to \mathbb{R}^{d'}$ is an isometry, for every $1 \le i \le \ell$ there exists $U_i' \in O(d')$ and $a_i' \in \mathbb{R}^{d'}$ so that

$$S \circ \pi_{\mathbb{V}} \circ \varphi_i \circ S^{-1}(x) = r_i U_i' x + a_i' \text{ for } x \in \mathbb{R}^{d'} .$$

In particular, $\{S \circ \pi_{\mathbb{V}} \circ \varphi_i \circ S^{-1}\}_{i=1}^{\ell}$ is a self-similar IFS on $\mathbb{R}^{d'}$.

Now we can state our main result, which extends the aforementioned theorems to higher dimensions.

Theorem (R., (2021))

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ *be an affinely irreducible self-similar IFS on* $\mathbb{R}^d$*.*

Theorem (R., (2021))

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ *be an affinely irreducible self-similar IFS on* $\mathbb{R}^d$*. Then there exists* $p \in \Delta_\ell$ *such that* μ_p *is non-Rajchman if and only if*

Theorem (R., (2021))

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ *be an affinely irreducible self-similar IFS on* $\mathbb{R}^d$*. Then there exists* $p \in \Delta_\ell$ *such that* μ_p *is non-Rajchman if and only if there exists a linear subspace* $\mathbb{V} \subset \mathbb{R}^d$*, with* $d' := \dim \mathbb{V} > 0$

Theorem (R., (2021))

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ *be an affinely irreducible self-similar IFS on* $\mathbb{R}^d$*. Then there exists* $p \in \Delta_\ell$ *such that* μ_p *is non-Rajchman if and only if there exists a linear subspace* $\mathbb{V} \subset \mathbb{R}^d$*, with* $d' := \dim \mathbb{V} > 0$ *and* $U_i(\mathbb{V}) = \mathbb{V}$ *for* $1 \leq i \leq \ell$*,*

Theorem (R., (2021))

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ *be an affinely irreducible self-similar IFS on* $\mathbb{R}^d$. *Then there exists* $p \in \Delta_\ell$ *such that* μ_p *is non-Rajchman if and only if there exists a linear subspace* $\mathbb{V} \subset \mathbb{R}^d$, *with* $d' := \dim \mathbb{V} > 0$ *and* $U_i(\mathbb{V}) = \mathbb{V}$ *for* $1 \le i \le \ell$, *and an isometry* $S : \mathbb{V} \to \mathbb{R}^{d'}$ *so that the following conditions are satisfied:*

Theorem (R., (2021))

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ *be an affinely irreducible self-similar IFS on* $\mathbb{R}^d$*. Then there exists* $p \in \Delta_\ell$ *such that* μ_p *is non-Rajchman if and only if there exists a linear subspace* $\mathbb{V} \subset \mathbb{R}^d$*, with* $d' := \dim \mathbb{V} > 0$ *and* $U_i(\mathbb{V}) = \mathbb{V}$ *for* $1 \le i \le \ell$*, and an isometry* $S : \mathbb{V} \to \mathbb{R}^{d'}$ *so that the following conditions are satisfied:*

1. *For* $1 \le i \le \ell$ *let* $U_i' \in O(d')$ *and* $a_i' \in \mathbb{R}^{d'}$ *be with* $S \circ \pi_{\mathbb{V}} \circ \varphi_i \circ S^{-1}(x) = r_i U_i' x + a_i'$.

Theorem (R., (2021))

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ *be an affinely irreducible self-similar IFS on* $\mathbb{R}^d$*. Then there exists* $p \in \Delta_\ell$ *such that* μ_p *is non-Rajchman if and only if there exists a linear subspace* $\mathbb{V} \subset \mathbb{R}^d$*, with* $d' := \dim \mathbb{V} > 0$ *and* $U_i(\mathbb{V}) = \mathbb{V}$ *for* $1 \le i \le \ell$*, and an isometry* $S : \mathbb{V} \to \mathbb{R}^{d'}$ *so that the following conditions are satisfied:*

1. *For* $1 \le i \le \ell$ *let* $U_i' \in O(d')$ *and* $a_i' \in \mathbb{R}^{d'}$ *be with* $S \circ \pi_{\mathbb{V}} \circ \varphi_i \circ S^{-1}(x) = r_i U_i' x + a_i'$*. Let* $\mathbf{H} \subset GL_{d'}(\mathbb{R})$ *be the group generated by* $\{r_i U_i'\}_{i=1}^{\ell}$*, and set* $\mathbf{N} := \mathbf{H} \cap O(d')$*.*

Theorem (R., (2021))

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ *be an affinely irreducible self-similar IFS on* $\mathbb{R}^d$. *Then there exists* $p \in \Delta_\ell$ *such that* μ_p *is non-Rajchman if and only if there exists a linear subspace* $\mathbb{V} \subset \mathbb{R}^d$, *with* $d' := \dim \mathbb{V} > 0$ *and* $U_i(\mathbb{V}) = \mathbb{V}$ *for* $1 \le i \le \ell$, *and an isometry* $S : \mathbb{V} \to \mathbb{R}^{d'}$ *so that the following conditions are satisfied:*

1. *For* $1 \le i \le \ell$ *let* $U_i' \in O(d')$ *and* $a_i' \in \mathbb{R}^{d'}$ *be with* $S \circ \pi_{\mathbb{V}} \circ \varphi_i \circ S^{-1}(x) = r_i U_i' x + a_i'$. *Let* $\mathbf{H} \subset GL_{d'}(\mathbb{R})$ *be the group generated by* $\{r_i U_i'\}_{i=1}^{\ell}$, *and set* $\mathbf{N} := \mathbf{H} \cap O(d')$. *Then* $\mathbf{N}$ *is finite,* $\mathbf{N} \triangleleft \mathbf{H}$ *and* $\mathbf{H}/\mathbf{N}$ *is cyclic.*

Theorem (R., (2021))

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ *be an affinely irreducible self-similar IFS on* $\mathbb{R}^d$*. Then there exists* $p \in \Delta_\ell$ *such that* μ_p *is non-Rajchman if and only if there exists a linear subspace* $\mathbb{V} \subset \mathbb{R}^d$*, with* $d' := \dim \mathbb{V} > 0$ *and* $U_i(\mathbb{V}) = \mathbb{V}$ *for* $1 \leq i \leq \ell$*, and an isometry* $S : \mathbb{V} \to \mathbb{R}^{d'}$ *so that the following conditions are satisfied:*

1. *For* $1 \leq i \leq \ell$ *let* $U_i' \in O(d')$ *and* $a_i' \in \mathbb{R}^{d'}$ *be with* $S \circ \pi_{\mathbb{V}} \circ \varphi_i \circ S^{-1}(x) = r_i U_i' x + a_i'$*. Let* $\mathbf{H} \subset GL_{d'}(\mathbb{R})$ *be the group generated by* $\{r_i U_i'\}_{i=1}^{\ell}$*, and set* $\mathbf{N} := \mathbf{H} \cap O(d')$*. Then* $\mathbf{N}$ *is finite,* $\mathbf{N} \triangleleft \mathbf{H}$ *and* $\mathbf{H}/\mathbf{N}$ *is cyclic.*
2. *For every contracting* $A \in \mathbf{H}$ *with* $\{A^n \mathbf{N}\}_{n \in \mathbb{Z}} = \mathbf{H}/\mathbf{N}$

Theorem (R., (2021))

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ *be an affinely irreducible self-similar IFS on* $\mathbb{R}^d$*. Then there exists* $p \in \Delta_\ell$ *such that* μ_p *is non-Rajchman if and only if there exists a linear subspace* $\mathbb{V} \subset \mathbb{R}^d$*, with* $d' := \dim \mathbb{V} > 0$ *and* $U_i(\mathbb{V}) = \mathbb{V}$ *for* $1 \le i \le \ell$*, and an isometry* $S : \mathbb{V} \to \mathbb{R}^{d'}$ *so that the following conditions are satisfied:*

1. *For* $1 \le i \le \ell$ *let* $U_i' \in O(d')$ *and* $a_i' \in \mathbb{R}^{d'}$ *be with* $S \circ \pi_{\mathbb{V}} \circ \varphi_i \circ S^{-1}(x) = r_i U_i' x + a_i'$*. Let* $\mathbf{H} \subset GL_{d'}(\mathbb{R})$ *be the group generated by* $\{r_i U_i'\}_{i=1}^{\ell}$*, and set* $\mathbf{N} := \mathbf{H} \cap O(d')$*. Then* $\mathbf{N}$ *is finite,* $\mathbf{N} \triangleleft \mathbf{H}$ *and* $\mathbf{H}/\mathbf{N}$ *is cyclic.*
2. *For every contracting* $A \in \mathbf{H}$ *with* $\{A^n \mathbf{N}\}_{n \in \mathbb{Z}} = \mathbf{H}/\mathbf{N}$ *there exist* $k \ge 1$*,* $\theta_1, ..., \theta_k \in \mathbb{C}$ *and* $\zeta_1, ..., \zeta_k \in \mathbb{C}^{d'} \setminus \{0\}$ *so that:*

Theorem (R., (2021))

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ be an affinely irreducible self-similar IFS on $\mathbb{R}^d$. Then there exists $p \in \Delta_\ell$ such that μ_p is non-Rajchman if and only if there exists a linear subspace $\mathbb{V} \subset \mathbb{R}^d$, with $d' := \dim \mathbb{V} > 0$ and $U_i(\mathbb{V}) = \mathbb{V}$ for $1 \le i \le \ell$, and an isometry $S : \mathbb{V} \to \mathbb{R}^{d'}$ so that the following conditions are satisfied:

1. *For $1 \le i \le \ell$ let $U_i' \in O(d')$ and $a_i' \in \mathbb{R}^{d'}$ be with $S \circ \pi_{\mathbb{V}} \circ \varphi_i \circ S^{-1}(x) = r_i U_i' x + a_i'$. Let $\mathbf{H} \subset GL_{d'}(\mathbb{R})$ be the group generated by $\{r_i U_i'\}_{i=1}^{\ell}$, and set $\mathbf{N} := \mathbf{H} \cap O(d')$. Then $\mathbf{N}$ is finite, $\mathbf{N} \lhd \mathbf{H}$ and $\mathbf{H}/\mathbf{N}$ is cyclic.*
2. *For every contracting $A \in \mathbf{H}$ with $\{A^n \mathbf{N}\}_{n \in \mathbb{Z}} = \mathbf{H}/\mathbf{N}$ there exist $k \ge 1$, $\theta_1, ..., \theta_k \in \mathbb{C}$ and $\zeta_1, ..., \zeta_k \in \mathbb{C}^{d'} \setminus \{0\}$ so that:*
   1. *$\{\theta_1, ..., \theta_k\}$ is a P.V. k-tuple;*

Theorem (R., (2021))

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ be an affinely irreducible self-similar IFS on $\mathbb{R}^d$. Then there exists $p \in \Delta_\ell$ such that μ_p is non-Rajchman if and only if there exists a linear subspace $\mathbb{V} \subset \mathbb{R}^d$, with $d' := \dim \mathbb{V} > 0$ and $U_i(\mathbb{V}) = \mathbb{V}$ for $1 \le i \le \ell$, and an isometry $S : \mathbb{V} \to \mathbb{R}^{d'}$ so that the following conditions are satisfied:

1. *For $1 \le i \le \ell$ let $U_i' \in O(d')$ and $a_i' \in \mathbb{R}^{d'}$ be with $S \circ \pi_{\mathbb{V}} \circ \varphi_i \circ S^{-1}(x) = r_i U_i' x + a_i'$. Let $\mathbf{H} \subset GL_{d'}(\mathbb{R})$ be the group generated by $\{r_i U_i'\}_{i=1}^{\ell}$, and set $\mathbf{N} := \mathbf{H} \cap O(d')$. Then $\mathbf{N}$ is finite, $\mathbf{N} \triangleleft \mathbf{H}$ and $\mathbf{H}/\mathbf{N}$ is cyclic.*
2. *For every contracting $A \in \mathbf{H}$ with $\{A^n \mathbf{N}\}_{n \in \mathbb{Z}} = \mathbf{H}/\mathbf{N}$ there exist $k \ge 1$, $\theta_1, ..., \theta_k \in \mathbb{C}$ and $\zeta_1, ..., \zeta_k \in \mathbb{C}^{d'} \setminus \{0\}$ so that:*
   - a. *$\{\theta_1, ..., \theta_k\}$ is a P.V. k-tuple;*
   - b. *$A^{-1}\zeta_j = \theta_j \zeta_j$ for $1 \le j \le k$;*

Theorem (R., (2021))

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ be an affinely irreducible self-similar IFS on $\mathbb{R}^d$. Then there exists $p \in \Delta_\ell$ such that μ_p is non-Rajchman if and only if there exists a linear subspace $\mathbb{V} \subset \mathbb{R}^d$, with $d' := \dim \mathbb{V} > 0$ and $U_i(\mathbb{V}) = \mathbb{V}$ for $1 \le i \le \ell$, and an isometry $S : \mathbb{V} \to \mathbb{R}^{d'}$ so that the following conditions are satisfied:

1. *For $1 \le i \le \ell$ let $U_i' \in O(d')$ and $a_i' \in \mathbb{R}^{d'}$ be with $S \circ \pi_{\mathbb{V}} \circ \varphi_i \circ S^{-1}(x) = r_i U_i' x + a_i'$. Let $\mathbf{H} \subset GL_{d'}(\mathbb{R})$ be the group generated by $\{r_i U_i'\}_{i=1}^{\ell}$, and set $\mathbf{N} := \mathbf{H} \cap O(d')$. Then $\mathbf{N}$ is finite, $\mathbf{N} \triangleleft \mathbf{H}$ and $\mathbf{H}/\mathbf{N}$ is cyclic.*
2. *For every contracting $A \in \mathbf{H}$ with $\{A^n \mathbf{N}\}_{n \in \mathbb{Z}} = \mathbf{H}/\mathbf{N}$ there exist $k \ge 1$, $\theta_1, ..., \theta_k \in \mathbb{C}$ and $\zeta_1, ..., \zeta_k \in \mathbb{C}^{d'} \setminus \{0\}$ so that:*
   1. *$\{\theta_1, ..., \theta_k\}$ is a P.V. k-tuple;*
   2. *$A^{-1}\zeta_j = \theta_j \zeta_j$ for $1 \le j \le k$;*
   3. *for every $1 \le i \le \ell$ and $V \in \mathbf{N}$ there exists $P_{i,V} \in \mathbb{Q}[X]$*

Theorem (R., (2021))

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ *be an affinely irreducible self-similar IFS on* $\mathbb{R}^d$*. Then there exists* $p \in \Delta_\ell$ *such that* μ_p *is non-Rajchman if and only if there exists a linear subspace* $\mathbb{V} \subset \mathbb{R}^d$*, with* $d' := \dim \mathbb{V} > 0$ *and* $U_i(\mathbb{V}) = \mathbb{V}$ *for* $1 \leq i \leq \ell$*, and an isometry* $S : \mathbb{V} \to \mathbb{R}^{d'}$ *so that the following conditions are satisfied:*

1. *For* $1 \leq i \leq \ell$ *let* $U_i' \in O(d')$ *and* $a_i' \in \mathbb{R}^{d'}$ *be with* $S \circ \pi_{\mathbb{V}} \circ \varphi_i \circ S^{-1}(x) = r_i U_i' x + a_i'$*. Let* $\mathbf{H} \subset GL_{d'}(\mathbb{R})$ *be the group generated by* $\{r_i U_i'\}_{i=1}^{\ell}$*, and set* $\mathbf{N} := \mathbf{H} \cap O(d')$*. Then* $\mathbf{N}$ *is finite,* $\mathbf{N} \triangleleft \mathbf{H}$ *and* $\mathbf{H}/\mathbf{N}$ *is cyclic.*
2. *For every contracting* $A \in \mathbf{H}$ *with* $\{A^n \mathbf{N}\}_{n \in \mathbb{Z}} = \mathbf{H}/\mathbf{N}$ *there exist* $k \geq 1$*,* $\theta_1, ..., \theta_k \in \mathbb{C}$ *and* $\zeta_1, ..., \zeta_k \in \mathbb{C}^{d'} \setminus \{0\}$ *so that:*
   - a. $\{\theta_1, ..., \theta_k\}$ *is a P.V.* k*-tuple;*
   - b. $A^{-1}\zeta_j = \theta_j \zeta_j$ *for* $1 \leq j \leq k$*;*
   - c. *for every* $1 \leq i \leq \ell$ *and* $V \in \mathbf{N}$ *there exists* $P_{i,V} \in \mathbb{Q}[X]$ *so that* $\langle V a_i', \zeta_j \rangle = P_{i,V}(\theta_j)$ *for* $1 \leq j \leq k$*.*

Remarks regarding the theorem

Remarks regarding the theorem

- When $d = 1$ and Φ is orientation preserving, the theorem is easily seen to be equivalent to the theorems of Li & Sahlsten and of Brémont.

Remarks regarding the theorem

- When $d = 1$ and Φ is orientation preserving, the theorem is easily seen to be equivalent to the theorems of Li & Sahlsten and of Brémont.
- In fact, if condition (1) holds and condition (2) is satisfied for **some** contracting $A \in \mathbf{H}$ with $\{A^n\mathbf{N}\}_{n\in\mathbb{Z}} = \mathbf{H}/\mathbf{N}$, then there exists $p \in \Delta_\ell$ so that μ_p is non-Rajchman.

Remarks regarding the theorem

- When $d = 1$ and Φ is orientation preserving, the theorem is easily seen to be equivalent to the theorems of Li & Sahlsten and of Brémont.
- In fact, if condition (1) holds and condition (2) is satisfied for **some** contracting $A \in \mathbf{H}$ with $\{A^n \mathbf{N}\}_{n \in \mathbb{Z}} = \mathbf{H}/\mathbf{N}$, then there exists $p \in \Delta_\ell$ so that μ_p is non-Rajchman.
- In condition (1), since $\mathbf{N}$ is the kernel of the homomorphism sending $rU \in \mathbf{H}$ to r, it is obvious that $\mathbf{N} \triangleleft \mathbf{H}$.

Remarks regarding the theorem

- When $d = 1$ and Φ is orientation preserving, the theorem is easily seen to be equivalent to the theorems of Li & Sahlsten and of Brémont.
- In fact, if condition (1) holds and condition (2) is satisfied for **some** contracting $A \in \mathbf{H}$ with $\{A^n\mathbf{N}\}_{n\in\mathbb{Z}} = \mathbf{H}/\mathbf{N}$, then there exists $p \in \Delta_\ell$ so that μ_p is non-Rajchman.
- In condition (1), since $\mathbf{N}$ is the kernel of the homomorphism sending $rU \in \mathbf{H}$ to r, it is obvious that $\mathbf{N} \lhd \mathbf{H}$. The statements regarding the finiteness of $\mathbf{N}$ and $\mathbf{H}/\mathbf{N}$ being cyclic are the interesting part of this condition.

Remarks regarding the theorem

- When $d = 1$ and Φ is orientation preserving, the theorem is easily seen to be equivalent to the theorems of Li & Sahlsten and of Brémont.
- In fact, if condition (1) holds and condition (2) is satisfied for **some** contracting $A \in \mathbf{H}$ with $\{A^n \mathbf{N}\}_{n \in \mathbb{Z}} = \mathbf{H}/\mathbf{N}$, then there exists $p \in \Delta_\ell$ so that μ_p is non-Rajchman.
- In condition (1), since $\mathbf{N}$ is the kernel of the homomorphism sending $rU \in \mathbf{H}$ to r, it is obvious that $\mathbf{N} \triangleleft \mathbf{H}$. The statements regarding the finiteness of $\mathbf{N}$ and $\mathbf{H}/\mathbf{N}$ being cyclic are the interesting part of this condition.
- The parameters k and $\theta_1, ..., \theta_k$ in condition (2) may depend on the choice of A.

Remarks regarding the theorem

- When $d = 1$ and Φ is orientation preserving, the theorem is easily seen to be equivalent to the theorems of Li & Sahlsten and of Brémont.
- In fact, if condition (1) holds and condition (2) is satisfied for **some** contracting $A \in \mathbf{H}$ with $\{A^n\mathbf{N}\}_{n\in\mathbb{Z}} = \mathbf{H}/\mathbf{N}$, then there exists $p \in \Delta_\ell$ so that μ_p is non-Rajchman.
- In condition (1), since $\mathbf{N}$ is the kernel of the homomorphism sending $rU \in \mathbf{H}$ to r, it is obvious that $\mathbf{N} \triangleleft \mathbf{H}$. The statements regarding the finiteness of $\mathbf{N}$ and $\mathbf{H}/\mathbf{N}$ being cyclic are the interesting part of this condition.
- The parameters k and $\theta_1, ..., \theta_k$ in condition (2) may depend on the choice of A.
- In condition (2), since A^{-1} is a member of $\mathbf{H}$ all of its eigenvalues have the same modulus. In particular $|\theta_1| = ... = |\theta_k|$.

- The theorem provides many explicit examples of affinely irreducible IFSs Φ for which there exists $p \in \Delta_\ell$ so that μ_p is non-Rajchman.

- The theorem provides many explicit examples of affinely irreducible IFSs Φ for which there exists $p \in \Delta_\ell$ so that μ_p is non-Rajchman. In fact, for every $k \geq 1$ and $\theta_1, ..., \theta_k \in \mathbb{C}$ such that $\{\theta_1, ..., \theta_k\}$ is a P.V. k-tuple and $|\theta_1| = ... = |\theta_k|$,

- The theorem provides many explicit examples of affinely irreducible IFSs Φ for which there exists $p \in \Delta_\ell$ so that μ_p is non-Rajchman. In fact, for every $k \geq 1$ and $\theta_1, ..., \theta_k \in \mathbb{C}$ such that $\{\theta_1, ..., \theta_k\}$ is a P.V. k-tuple and $|\theta_1| = ... = |\theta_k|$, it is possible to construct a corresponding IFS on $\mathbb{R}^k$ with these properties.

- The theorem provides many explicit examples of affinely irreducible IFSs Φ for which there exists $p \in \Delta_\ell$ so that μ_p is non-Rajchman. In fact, for every $k \geq 1$ and $\theta_1, ..., \theta_k \in \mathbb{C}$ such that $\{\theta_1, ..., \theta_k\}$ is a P.V. k-tuple and $|\theta_1| = ... = |\theta_k|$, it is possible to construct a corresponding IFS on $\mathbb{R}^k$ with these properties.
- The theorem can be used to verify the Rajchman property in many situations.

- The theorem provides many explicit examples of affinely irreducible IFSs Φ for which there exists $p \in \Delta_\ell$ so that μ_p is non-Rajchman. In fact, for every $k \geq 1$ and $\theta_1, ..., \theta_k \in \mathbb{C}$ such that $\{\theta_1, ..., \theta_k\}$ is a P.V. k-tuple and $|\theta_1| = ... = |\theta_k|$, it is possible to construct a corresponding IFS on $\mathbb{R}^k$ with these properties.
- The theorem can be used to verify the Rajchman property in many situations. For instance we have the following corollary, which follows almost directly from the theorem.

- The theorem provides many explicit examples of affinely irreducible IFSs Φ for which there exists $p \in \Delta_\ell$ so that μ_p is non-Rajchman. In fact, for every $k \geq 1$ and $\theta_1, ..., \theta_k \in \mathbb{C}$ such that $\{\theta_1, ..., \theta_k\}$ is a P.V. k-tuple and $|\theta_1| = ... = |\theta_k|$, it is possible to construct a corresponding IFS on $\mathbb{R}^k$ with these properties.
- The theorem can be used to verify the Rajchman property in many situations. For instance we have the following corollary, which follows almost directly from the theorem.

Corollary

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ *be an affinely irreducible self-similar IFS on* $\mathbb{R}^d$*.*

- The theorem provides many explicit examples of affinely irreducible IFSs Φ for which there exists $p \in \Delta_\ell$ so that μ_p is non-Rajchman. In fact, for every $k \geq 1$ and $\theta_1, ..., \theta_k \in \mathbb{C}$ such that $\{\theta_1, ..., \theta_k\}$ is a P.V. k-tuple and $|\theta_1| = ... = |\theta_k|$, it is possible to construct a corresponding IFS on $\mathbb{R}^k$ with these properties.
- The theorem can be used to verify the Rajchman property in many situations. For instance we have the following corollary, which follows almost directly from the theorem.

Corollary

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ be an affinely irreducible self-similar IFS on $\mathbb{R}^d$. Suppose that there exists $p \in \Delta_\ell$ so that μ_p is non-Rajchman.

- The theorem provides many explicit examples of affinely irreducible IFSs Φ for which there exists $p \in \Delta_\ell$ so that μ_p is non-Rajchman. In fact, for every $k \geq 1$ and $\theta_1, ..., \theta_k \in \mathbb{C}$ such that $\{\theta_1, ..., \theta_k\}$ is a P.V. k-tuple and $|\theta_1| = ... = |\theta_k|$, it is possible to construct a corresponding IFS on $\mathbb{R}^k$ with these properties.
- The theorem can be used to verify the Rajchman property in many situations. For instance we have the following corollary, which follows almost directly from the theorem.

Corollary

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ be an affinely irreducible self-similar IFS on $\mathbb{R}^d$. Suppose that there exists $p \in \Delta_\ell$ so that μ_p is non-Rajchman. Then there exists an algebraic integer $\theta > 1$ so that for every $1 \leq i \leq \ell$ there exists a rational integer $n_i \geq 1$ with $r_i = \theta^{-n_i}$.

Ideas from the proof of the theorem

Ideas from the proof of the theorem

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ be an affinely irreducible self-similar IFS on $\mathbb{R}^d$.

Ideas from the proof of the theorem

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ be an affinely irreducible self-similar IFS on $\mathbb{R}^d$. Suppose there exists $(p_i)_{i=1}^{\ell} = p \in \Delta_\ell$ so that $\mu := \mu_p$ is non-Rajchman.

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ be an affinely irreducible self-similar IFS on $\mathbb{R}^d$. Suppose there exists $(p_i)_{i=1}^{\ell} = p \in \Delta_\ell$ so that $\mu := \mu_p$ is non-Rajchman.

Write $G \subset \mathbb{R} \times O(d)$ for the closed subgroup generated by $\{(\log_2 r_i^{-1}, U_i)\}_{i=1}^{\ell}$.

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ be an affinely irreducible self-similar IFS on $\mathbb{R}^d$. Suppose there exists $(p_i)_{i=1}^{\ell} = p \in \Delta_\ell$ so that $\mu := \mu_p$ is non-Rajchman.

Write $G \subset \mathbb{R} \times O(d)$ for the closed subgroup generated by $\{(\log_2 r_i^{-1}, U_i)\}_{i=1}^{\ell}$.

Let $\psi : G \to \mathbb{R}$ be the homomorphism with $\psi(t, U) = t$.

Ideas from the proof of the theorem

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ be an affinely irreducible self-similar IFS on $\mathbb{R}^d$. Suppose there exists $(p_i)_{i=1}^{\ell} = p \in \Delta_\ell$ so that $\mu := \mu_p$ is non-Rajchman.

Write $G \subset \mathbb{R} \times O(d)$ for the closed subgroup generated by $\{(\log_2 r_i^{-1}, U_i)\}_{i=1}^{\ell}$.

Let $\psi : G \to \mathbb{R}$ be the homomorphism with $\psi(t, U) = t$. Note that $\psi(G)$ is the closed subgroup of $\mathbb{R}$ generated by $\{\log_2 r_i\}_{i=1}^{\ell}$.

Ideas from the proof of the theorem

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ be an affinely irreducible self-similar IFS on $\mathbb{R}^d$. Suppose there exists $(p_i)_{i=1}^{\ell} = p \in \Delta_\ell$ so that $\mu := \mu_p$ is non-Rajchman.

Write $G \subset \mathbb{R} \times O(d)$ for the closed subgroup generated by $\{(\log_2 r_i^{-1}, U_i)\}_{i=1}^{\ell}$.

Let $\psi : G \to \mathbb{R}$ be the homomorphism with $\psi(t, U) = t$. Note that $\psi(G)$ is the closed subgroup of $\mathbb{R}$ generated by $\{\log_2 r_i\}_{i=1}^{\ell}$.

Let $\gamma : \psi(G) \to G$ be a continuous homomorphism with $\psi \circ \gamma = Id$.

Ideas from the proof of the theorem

Let $\Phi = \{\varphi_i(x) = r_i U_i x + a_i\}_{i=1}^{\ell}$ be an affinely irreducible self-similar IFS on $\mathbb{R}^d$. Suppose there exists $(p_i)_{i=1}^{\ell} = p \in \Delta_\ell$ so that $\mu := \mu_p$ is non-Rajchman.

Write $G \subset \mathbb{R} \times O(d)$ for the closed subgroup generated by $\{(\log_2 r_i^{-1}, U_i)\}_{i=1}^{\ell}$.

Let $\psi : G \to \mathbb{R}$ be the homomorphism with $\psi(t, U) = t$. Note that $\psi(G)$ is the closed subgroup of $\mathbb{R}$ generated by $\{\log_2 r_i\}_{i=1}^{\ell}$.

Let $\gamma : \psi(G) \to G$ be a continuous homomorphism with $\psi \circ \gamma = Id$. We write γ_t in place of $\gamma(t)$.

Let $X_1, X_2, ...$ be i.i.d. G-valued random elements with

$$\mathbb{P}\{X_1 = (\log_2 r_i^{-1}, U_i)\} = p_i \text{ for } 1 \le i \le \ell,$$

Let $X_1, X_2, \ldots$ be i.i.d. G-valued random elements with

$$\mathbb{P}\{X_1 = (\log_2 r_i^{-1}, U_i)\} = p_i \text{ for } 1 \le i \le \ell,$$

and for $n \ge 1$ set $Y_n := X_1 \cdot \ldots \cdot X_n$.

Let $X_1, X_2, ...$ be i.i.d. G-valued random elements with

$$\mathbb{P}\{X_1 = (\log_2 r_i^{-1}, U_i)\} = p_i \text{ for } 1 \leq i \leq \ell,$$

and for $n \geq 1$ set $Y_n := X_1 \cdot ... \cdot X_n$.

For $t > 0$ set $\tau_t := \min\{n \geq 1 \, : \, \psi(Y_n) \geq t\}$.

Let $X_1, X_2, \ldots$ be i.i.d. G-valued random elements with

$$\mathbb{P}\{X_1 = (\log_2 r_i^{-1}, U_i)\} = p_i \text{ for } 1 \le i \le \ell,$$

and for $n \ge 1$ set $Y_n := X_1 \cdot \ldots \cdot X_n$.

For $t > 0$ set $\tau_t := \min\{n \ge 1 \,:\, \psi(Y_n) \ge t\}$.

The following statement, which extends the classical renewal theorem, is essentially due to Y. Guivarc'h.

Let $X_1, X_2, ...$ be i.i.d. G-valued random elements with

$$\mathbb{P}\{X_1 = (\log_2 r_i^{-1}, U_i)\} = p_i \text{ for } 1 \leq i \leq \ell,$$

and for $n \geq 1$ set $Y_n := X_1 \cdot ... \cdot X_n$.

For $t > 0$ set $\tau_t := \min\{n \geq 1 \, : \, \psi(Y_n) \geq t\}$.

The following statement, which extends the classical renewal theorem, is essentially due to Y. Guivarc'h.

Theorem

As $t \to \infty$ the random elements $\gamma_{-t} Y_{\tau_t}$ converge in distribution to a probability measure ν on G which is absolutely continuous with respect to the Haar measure of G.

We use the last statement several times in our arguments.

We use the last statement several times in our arguments. In particular we use it to prove the following lemma, which is inspired by the work of Li & Sahlsten concerning the nondiscrete case on the line.

We use the last statement several times in our arguments. In particular we use it to prove the following lemma, which is inspired by the work of Li & Sahlsten concerning the nondiscrete case on the line. For $(t, U) \in G$ and $x \in \mathbb{R}^d$ set $(t, U).x := 2^t U x$.

We use the last statement several times in our arguments. In particular we use it to prove the following lemma, which is inspired by the work of Li & Sahlsten concerning the nondiscrete case on the line. For $(t, U) \in G$ and $x \in \mathbb{R}^d$ set $(t, U).x := 2^t U x$.

Lemma 1

For every $\epsilon > 0$ there exists $T > 1$ such that the following holds.

We use the last statement several times in our arguments. In particular we use it to prove the following lemma, which is inspired by the work of Li & Sahlsten concerning the nondiscrete case on the line. For $(t, U) \in G$ and $x \in \mathbb{R}^d$ set $(t, U).x := 2^t U x$.

Lemma 1

For every $\epsilon > 0$ there exists $T > 1$ such that the following holds. Let $t \geq T$ be with $t \in \psi(G)$ and let $\xi \in \mathbb{R}^d$ be with $|\xi| \leq \epsilon^{-1}$, then

We use the last statement several times in our arguments. In particular we use it to prove the following lemma, which is inspired by the work of Li & Sahlsten concerning the nondiscrete case on the line. For $(t, U) \in G$ and $x \in \mathbb{R}^d$ set $(t, U).x := 2^t U x$.

Lemma 1

For every $\epsilon > 0$ there exists $T > 1$ such that the following holds. Let $t \geq T$ be with $t \in \psi(G)$ and let $\xi \in \mathbb{R}^d$ be with $|\xi| \leq \epsilon^{-1}$, then

$$|\widehat{\mu}(\gamma_t.\xi)|^2 \leq \epsilon + \int \int \left| \int e^{i\langle g.\xi, x-y \rangle} d\nu(g) \right| d\mu(x)\, d\mu(y)\, .$$

We use the last statement several times in our arguments. In particular we use it to prove the following lemma, which is inspired by the work of Li & Sahlsten concerning the nondiscrete case on the line. For $(t, U) \in G$ and $x \in \mathbb{R}^d$ set $(t, U).x := 2^t U x$.

Lemma 1

For every $\epsilon > 0$ there exists $T > 1$ such that the following holds. Let $t \geq T$ be with $t \in \psi(G)$ and let $\xi \in \mathbb{R}^d$ be with $|\xi| \leq \epsilon^{-1}$, then

$$|\widehat{\mu}(\gamma_t.\xi)|^2 \leq \epsilon + \int \int \left| \int e^{i\langle g.\xi, x-y \rangle} d\nu(g) \right| d\mu(x)\, d\mu(y)\, .$$

This lemma is useful only when the group G is nondiscrete.

The next step of the argument requires the affine irreducibility of Φ.

The next step of the argument requires the affine irreducibility of Φ. From that assumption it follows easily that $\mu(\mathbb{V}) = 0$ for every proper affine subspace $\mathbb{V}$ of $\mathbb{R}^d$.

The next step of the argument requires the affine irreducibility of Φ. From that assumption it follows easily that $\mu(\mathbb{V}) = 0$ for every proper affine subspace $\mathbb{V}$ of $\mathbb{R}^d$. The proof of the following lemma relies on this fact.

The next step of the argument requires the affine irreducibility of Φ. From that assumption it follows easily that $\mu(\mathbb{V}) = 0$ for every proper affine subspace $\mathbb{V}$ of $\mathbb{R}^d$. The proof of the following lemma relies on this fact.

Lemma 2

For every $\epsilon > 0$ there exists $S > 1$ so that the following holds.

The next step of the argument requires the affine irreducibility of Φ. From that assumption it follows easily that $\mu(\mathbb{V}) = 0$ for every proper affine subspace $\mathbb{V}$ of $\mathbb{R}^d$. The proof of the following lemma relies on this fact.

Lemma 2

For every $\epsilon > 0$ there exists $S > 1$ so that the following holds. Let $s \geq S$ and let $c : [0, 1] \to \mathbb{R}^d$ be a smooth curve with $|c'(t)| \geq \epsilon$ and $|c''(t)| \leq \epsilon^{-1}$ for all $0 \leq t \leq 1$,

The next step of the argument requires the affine irreducibility of Φ. From that assumption it follows easily that $\mu(\mathbb{V}) = 0$ for every proper affine subspace $\mathbb{V}$ of $\mathbb{R}^d$. The proof of the following lemma relies on this fact.

Lemma 2

For every $\epsilon > 0$ there exists $S > 1$ so that the following holds. Let $s \geq S$ and let $c : [0,1] \to \mathbb{R}^d$ be a smooth curve with $|c'(t)| \geq \epsilon$ and $|c''(t)| \leq \epsilon^{-1}$ for all $0 \leq t \leq 1$, then

$$\int \int \left| \int_0^1 e^{is\langle c(t), x-y \rangle} \, dt \right| \, d\mu(x) \, d\mu(y) < \epsilon \, .$$

Now it is not difficult to show that $\psi(G) \neq \mathbb{R}$.

Now it is not difficult to show that $\psi(G) \neq \mathbb{R}$. Assume by contradiction that this is not the case.

Now it is not difficult to show that $\psi(G) \neq \mathbb{R}$. Assume by contradiction that this is not the case.

Set $N := \ker \psi$,

Now it is not difficult to show that $\psi(G) \neq \mathbb{R}$. Assume by contradiction that this is not the case.

Set $N := \ker \psi$, note that N is a compact subgroup of G,

Now it is not difficult to show that $\psi(G) \neq \mathbb{R}$. Assume by contradiction that this is not the case.

Set $N := \ker \psi$, note that N is a compact subgroup of G, and let $\mathbf{m}_N$ be the Haar measure of N.

Now it is not difficult to show that $\psi(G) \neq \mathbb{R}$. Assume by contradiction that this is not the case.

Set $N := \ker \psi$, note that N is a compact subgroup of G, and let $\mathbf{m}_N$ be the Haar measure of N.

Since $\psi(G) = \mathbb{R}$ and by the definition of ν,

Now it is not difficult to show that $\psi(G) \neq \mathbb{R}$. Assume by contradiction that this is not the case.

Set $N := \ker \psi$, note that N is a compact subgroup of G, and let $\mathbf{m}_N$ be the Haar measure of N.

Since $\psi(G) = \mathbb{R}$ and by the definition of ν, it follows that there exists a piecewise constant $\rho : \mathbb{R} \to [0, \infty)$ so that

Now it is not difficult to show that $\psi(G) \neq \mathbb{R}$. Assume by contradiction that this is not the case.

Set $N := \ker \psi$, note that N is a compact subgroup of G, and let $\mathbf{m}_N$ be the Haar measure of N.

Since $\psi(G) = \mathbb{R}$ and by the definition of ν, it follows that there exists a piecewise constant $\rho : \mathbb{R} \to [0, \infty)$ so that

$$\int f \, d\nu = \int \int f(\gamma_t n) \rho(t) \, dt \, d\mathbf{m}_N(n) \textit{ for all } f \in C(G).$$

Now it is not difficult to show that $\psi(G) \neq \mathbb{R}$. Assume by contradiction that this is not the case.

Set $N := \ker \psi$, note that N is a compact subgroup of G, and let $\mathbf{m}_N$ be the Haar measure of N.

Since $\psi(G) = \mathbb{R}$ and by the definition of ν, it follows that there exists a piecewise constant $\rho : \mathbb{R} \to [0, \infty)$ so that

$$\int f \, d\nu = \int \int f(\gamma_t n) \rho(t) \, dt \, d\mathbf{m}_N(n) \textit{ for all } f \in C(G).$$

By using this decomposition together with Lemma 1,

Now it is not difficult to show that $\psi(G) \neq \mathbb{R}$. Assume by contradiction that this is not the case.

Set $N := \ker \psi$, note that N is a compact subgroup of G, and let $\mathbf{m}_N$ be the Haar measure of N.

Since $\psi(G) = \mathbb{R}$ and by the definition of ν, it follows that there exists a piecewise constant $\rho : \mathbb{R} \to [0, \infty)$ so that

$$\int f \, d\nu = \int \int f(\gamma_t n)\rho(t) \, dt \, d\mathbf{m}_N(n) \textit{ for all } f \in C(G).$$

By using this decomposition together with Lemma 1, and by applying Lemma 2 with curves c of the form $c(t) = \gamma_{ta+b} n.\xi$, where $a, b \in \mathbb{R}$, $n \in N$ and $\xi \in \mathbb{R}^d$,

Now it is not difficult to show that $\psi(G) \neq \mathbb{R}$. Assume by contradiction that this is not the case.

Set $N := \ker \psi$, note that N is a compact subgroup of G, and let $\mathbf{m}_N$ be the Haar measure of N.

Since $\psi(G) = \mathbb{R}$ and by the definition of ν, it follows that there exists a piecewise constant $\rho : \mathbb{R} \to [0, \infty)$ so that

$$\int f \, d\nu = \int \int f(\gamma_t n) \rho(t) \, dt \, d\mathbf{m}_N(n) \text{ for all } f \in C(G).$$

By using this decomposition together with Lemma 1, and by applying Lemma 2 with curves c of the form $c(t) = \gamma_{ta+b} n.\xi$, where $a, b \in \mathbb{R}$, $n \in N$ and $\xi \in \mathbb{R}^d$, we get that μ is Rajchman which contradicts our assumption.

Now it is not difficult to show that $\psi(G) \neq \mathbb{R}$. Assume by contradiction that this is not the case.

Set $N := \ker \psi$, note that N is a compact subgroup of G, and let $\mathbf{m}_N$ be the Haar measure of N.

Since $\psi(G) = \mathbb{R}$ and by the definition of ν, it follows that there exists a piecewise constant $\rho : \mathbb{R} \to [0, \infty)$ so that

$$\int f \, d\nu = \int \int f(\gamma_t n)\rho(t) \, dt \, d\mathbf{m}_N(n) \text{ for all } f \in C(G).$$

By using this decomposition together with Lemma 1, and by applying Lemma 2 with curves c of the form $c(t) = \gamma_{ta+b} n.\xi$, where $a, b \in \mathbb{R}$, $n \in N$ and $\xi \in \mathbb{R}^d$, we get that μ is Rajchman which contradicts our assumption. Thus, we must have $\psi(G) \neq \mathbb{R}$.

Next we want to make a reduction from the case in which G is nondiscrete with $\psi(G) \neq \mathbb{R}$, to the case in which G is discrete.

Next we want to make a reduction from the case in which G is nondiscrete with $\psi(G) \neq \mathbb{R}$, to the case in which G is discrete.

For this we need to choose appropriately the subspace $\mathbb{V} \subset \mathbb{R}^d$ appearing in the statement of the main theorem.

Next we want to make a reduction from the case in which G is nondiscrete with $\psi(G) \neq \mathbb{R}$, to the case in which G is discrete.

For this we need to choose appropriately the subspace $\mathbb{V} \subset \mathbb{R}^d$ appearing in the statement of the main theorem.

Denote by G_0 the connected component of G containing the identity element.

Next we want to make a reduction from the case in which G is nondiscrete with $\psi(G) \neq \mathbb{R}$, to the case in which G is discrete.

For this we need to choose appropriately the subspace $\mathbb{V} \subset \mathbb{R}^d$ appearing in the statement of the main theorem.

Denote by G_0 the connected component of G containing the identity element. We choose $\mathbb{V}$ to be the linear subspace consisting of all $x \in \mathbb{R}^d$ so that $g.x = x$ for all $g \in G_0$.

Next we want to make a reduction from the case in which G is nondiscrete with $\psi(G) \neq \mathbb{R}$, to the case in which G is discrete.

For this we need to choose appropriately the subspace $\mathbb{V} \subset \mathbb{R}^d$ appearing in the statement of the main theorem.

Denote by G_0 the connected component of G containing the identity element. We choose $\mathbb{V}$ to be the linear subspace consisting of all $x \in \mathbb{R}^d$ so that $g.x = x$ for all $g \in G_0$.

From $G_0 \triangleleft G$ it follows that $g.x \in \mathbb{V}$ for all $x \in \mathbb{V}$ and $g \in G$,

Next we want to make a reduction from the case in which G is nondiscrete with $\psi(G) \neq \mathbb{R}$, to the case in which G is discrete.

For this we need to choose appropriately the subspace $\mathbb{V} \subset \mathbb{R}^d$ appearing in the statement of the main theorem.

Denote by G_0 the connected component of G containing the identity element. We choose $\mathbb{V}$ to be the linear subspace consisting of all $x \in \mathbb{R}^d$ so that $g.x = x$ for all $g \in G_0$.

From $G_0 \triangleleft G$ it follows that $g.x \in \mathbb{V}$ for all $x \in \mathbb{V}$ and $g \in G$, which implies that $U_i(\mathbb{V}) = \mathbb{V}$ for $1 \leq i \leq \ell$.

Note that from $\psi(G) \neq \mathbb{R}$ it follows that G_0 is a compact Lie group.

Note that from $\psi(G) \neq \mathbb{R}$ it follows that G_0 is a compact Lie group. By using this fact, by representing ν as an average of certain smooth 1-dimensional measures, and by applying Lemmas 1 and 2 once more, we can prove the following proposition.

Note that from $\psi(G) \neq \mathbb{R}$ it follows that G_0 is a compact Lie group. By using this fact, by representing ν as an average of certain smooth 1-dimensional measures, and by applying Lemmas 1 and 2 once more, we can prove the following proposition. It enables us to perform the aforementioned reduction.

Note that from $\psi(G) \neq \mathbb{R}$ it follows that G_0 is a compact Lie group. By using this fact, by representing ν as an average of certain smooth 1-dimensional measures, and by applying Lemmas 1 and 2 once more, we can prove the following proposition. It enables us to perform the aforementioned reduction.

Proposition

For every $\epsilon > 0$ there exists $R > 1$ so that $|\widehat{\mu}(\xi)| < \epsilon$ for every $\xi \in \mathbb{R}^d$ with $|\pi_{\mathbb{V}^\perp}\xi| \geq \max\{R, \epsilon|\pi_{\mathbb{V}}\xi|\}$.

Note that from $\psi(G) \neq \mathbb{R}$ it follows that G_0 is a compact Lie group. By using this fact, by representing ν as an average of certain smooth 1-dimensional measures, and by applying Lemmas 1 and 2 once more, we can prove the following proposition. It enables us to perform the aforementioned reduction.

Proposition

For every $\epsilon > 0$ there exists $R > 1$ so that $|\widehat{\mu}(\xi)| < \epsilon$ for every $\xi \in \mathbb{R}^d$ with $|\pi_{\mathbb{V}^\perp}\xi| \geq \max\{R, \epsilon|\pi_{\mathbb{V}}\xi|\}$.

Observe that from the proposition and since μ is not Rajchman it follows that $d' := \dim \mathbb{V} > 0$.

Let $S : \mathbb{V} \to \mathbb{R}^{d'}$ be an isometry.

Let $S : \mathbb{V} \to \mathbb{R}^{d'}$ be an isometry. By using the last proposition, the fact that μ is not Rajchman, and the self-similarity of μ, we can show that $S\pi_{\mathbb{V}}\mu$ is also not Rajchman.

Let $S : \mathbb{V} \to \mathbb{R}^{d'}$ be an isometry. By using the last proposition, the fact that μ is not Rajchman, and the self-similarity of μ, we can show that $S\pi_{\mathbb{V}}\mu$ is also not Rajchman.

Note that the measure $S\pi_{\mathbb{V}}\mu$ is the self-similar measure corresponding to the IFS $\Phi' := \{S \circ \pi_{\mathbb{V}} \circ \varphi_i \circ S^{-1}\}_{i=1}^{\ell}$ and the probability vector p.

Let $S : \mathbb{V} \to \mathbb{R}^{d'}$ be an isometry. By using the last proposition, the fact that μ is not Rajchman, and the self-similarity of μ, we can show that $S\pi_{\mathbb{V}}\mu$ is also not Rajchman.

Note that the measure $S\pi_{\mathbb{V}}\mu$ is the self-similar measure corresponding to the IFS $\Phi' := \{S \circ \pi_{\mathbb{V}} \circ \varphi_i \circ S^{-1}\}_{i=1}^{\ell}$ and the probability vector p.

By using our choice of $\mathbb{V}$ it is not hard to show that the closed group generated by the linear parts of Φ' is discrete.

Let $S : \mathbb{V} \to \mathbb{R}^{d'}$ be an isometry. By using the last proposition, the fact that μ is not Rajchman, and the self-similarity of μ, we can show that $S\pi_{\mathbb{V}}\mu$ is also not Rajchman.

Note that the measure $S\pi_{\mathbb{V}}\mu$ is the self-similar measure corresponding to the IFS $\Phi' := \{S \circ \pi_{\mathbb{V}} \circ \varphi_i \circ S^{-1}\}_{i=1}^{\ell}$ and the probability vector p.

By using our choice of $\mathbb{V}$ it is not hard to show that the closed group generated by the linear parts of Φ' is discrete. Thus we have reduced the situation to the case in which Φ is discrete.

The treatment of the discrete case extends an argument due to Péter Varjú and Han Yu, who considered the case $d = 1$.

The treatment of the discrete case extends an argument due to Péter Varjú and Han Yu, who considered the case $d = 1$. One of the main ingredients is the following theorem.

The treatment of the discrete case extends an argument due to Péter Varjú and Han Yu, who considered the case $d = 1$. One of the main ingredients is the following theorem.

Theorem (C. Pisot, 1938)

Let $k \geq 1$ *and* $\theta_1, ..., \theta_k, \lambda_1, ..., \lambda_k \in \mathbb{C}$ *be with* $|\theta_j| > 1$ *and* $\lambda_j \neq 0$ *for* $1 \leq j \leq k$, *and* $\theta_j \neq \theta_i$ *for* $1 \leq j < i \leq k$.

The treatment of the discrete case extends an argument due to Péter Varjú and Han Yu, who considered the case $d = 1$. One of the main ingredients is the following theorem.

Theorem (C. Pisot, 1938)

Let $k \geq 1$ and $\theta_1, ..., \theta_k, \lambda_1, ..., \lambda_k \in \mathbb{C}$ be with $|\theta_j| > 1$ and $\lambda_j \neq 0$ for $1 \leq j \leq k$, and $\theta_j \neq \theta_i$ for $1 \leq j < i \leq k$. For $n \geq 0$ set $\eta_n = \sum_{j=1}^{k} \lambda_j \theta_j^n$, and suppose that $\eta_n \in \mathbb{R}$ for all $n \geq 0$.

The treatment of the discrete case extends an argument due to Péter Varjú and Han Yu, who considered the case $d = 1$. One of the main ingredients is the following theorem.

Theorem (C. Pisot, 1938)

Let $k \geq 1$ and $\theta_1, ..., \theta_k, \lambda_1, ..., \lambda_k \in \mathbb{C}$ be with $|\theta_j| > 1$ and $\lambda_j \neq 0$ for $1 \leq j \leq k$, and $\theta_j \neq \theta_i$ for $1 \leq j < i \leq k$. For $n \geq 0$ set $\eta_n = \sum_{j=1}^{k} \lambda_j \theta_j^n$, and suppose that $\eta_n \in \mathbb{R}$ for all $n \geq 0$. Moreover assume that $\sum_{n \geq 0} \|\eta_n\|^2 < \infty$.

The treatment of the discrete case extends an argument due to Péter Varjú and Han Yu, who considered the case $d = 1$. One of the main ingredients is the following theorem.

Theorem (C. Pisot, 1938)

Let $k \geq 1$ and $\theta_1, ..., \theta_k, \lambda_1, ..., \lambda_k \in \mathbb{C}$ be with $|\theta_j| > 1$ and $\lambda_j \neq 0$ for $1 \leq j \leq k$, and $\theta_j \neq \theta_i$ for $1 \leq j < i \leq k$. For $n \geq 0$ set $\eta_n = \sum_{j=1}^{k} \lambda_j \theta_j^n$, and suppose that $\eta_n \in \mathbb{R}$ for all $n \geq 0$. Moreover assume that $\sum_{n \geq 0} \|\eta_n\|^2 < \infty$. Then,

The treatment of the discrete case extends an argument due to Péter Varjú and Han Yu, who considered the case $d = 1$. One of the main ingredients is the following theorem.

Theorem (C. Pisot, 1938)

Let $k \geq 1$ and $\theta_1, ..., \theta_k, \lambda_1, ..., \lambda_k \in \mathbb{C}$ be with $|\theta_j| > 1$ and $\lambda_j \neq 0$ for $1 \leq j \leq k$, and $\theta_j \neq \theta_i$ for $1 \leq j < i \leq k$. For $n \geq 0$ set $\eta_n = \sum_{j=1}^{k} \lambda_j \theta_j^n$, and suppose that $\eta_n \in \mathbb{R}$ for all $n \geq 0$. Moreover assume that $\sum_{n \geq 0} \|\eta_n\|^2 < \infty$. Then,

1. *$\{\theta_1, ..., \theta_k\}$ is a P.V. k-tuple;*

The treatment of the discrete case extends an argument due to Péter Varjú and Han Yu, who considered the case $d = 1$. One of the main ingredients is the following theorem.

Theorem (C. Pisot, 1938)

Let $k \geq 1$ and $\theta_1, ..., \theta_k, \lambda_1, ..., \lambda_k \in \mathbb{C}$ be with $|\theta_j| > 1$ and $\lambda_j \neq 0$ for $1 \leq j \leq k$, and $\theta_j \neq \theta_i$ for $1 \leq j < i \leq k$. For $n \geq 0$ set $\eta_n = \sum_{j=1}^{k} \lambda_j \theta_j^n$, and suppose that $\eta_n \in \mathbb{R}$ for all $n \geq 0$. Moreover assume that $\sum_{n \geq 0} \|\eta_n\|^2 < \infty$. Then,

1. *$\{\theta_1, ..., \theta_k\}$ is a P.V. k-tuple;*
2. *$\lambda_j \in \mathbb{Q}(\theta_j)$ for each $1 \leq j \leq k$;*

The treatment of the discrete case extends an argument due to Péter Varjú and Han Yu, who considered the case $d = 1$. One of the main ingredients is the following theorem.

Theorem (C. Pisot, 1938)

Let $k \geq 1$ and $\theta_1, ..., \theta_k, \lambda_1, ..., \lambda_k \in \mathbb{C}$ be with $|\theta_j| > 1$ and $\lambda_j \neq 0$ for $1 \leq j \leq k$, and $\theta_j \neq \theta_i$ for $1 \leq j < i \leq k$. For $n \geq 0$ set $\eta_n = \sum_{j=1}^{k} \lambda_j \theta_j^n$, and suppose that $\eta_n \in \mathbb{R}$ for all $n \geq 0$. Moreover assume that $\sum_{n \geq 0} \|\eta_n\|^2 < \infty$. Then,

1. *$\{\theta_1, ..., \theta_k\}$ is a P.V. k-tuple;*
2. *$\lambda_j \in \mathbb{Q}(\theta_j)$ for each $1 \leq j \leq k$;*
3. *if $1 \leq j, i \leq k$ are such that θ_j and θ_i are conjugates over $\mathbb{Q}$ and $\sigma : \mathbb{Q}(\theta_j) \to \mathbb{Q}(\theta_i)$ is an isomorphism with $\sigma(\theta_j) = \theta_i$, then $\sigma(\lambda_j) = \lambda_i$.*

Thank You!