Hausdorff dimension of Besicovitch sets of Cantor graphs

Kornélia Héra

Alfréd Rényi Institute of Mathematics, Budapest, Hungary

Multifractal analysis and self-similarity

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joint work with Iqra Altaf and Marianna Csörnyei

History - Classical results

Definition

1. $B \subset \mathbb{R}^2$ is a Besicovitch set (or a Kakeya set) if it contains a unit line segment in every direction.
2. $K \subset \mathbb{R}^2$ is a Kakeya needle set if a unit line segment can be continuously turned around through 180 degrees within it.

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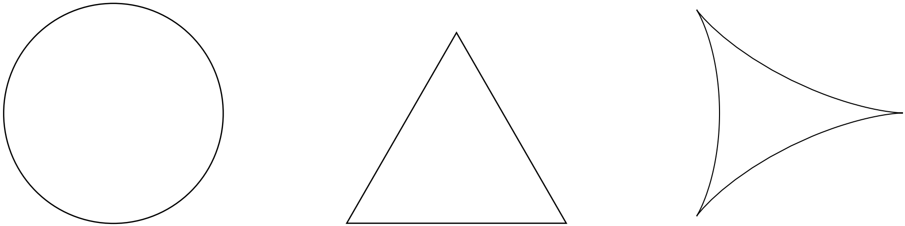
Examples



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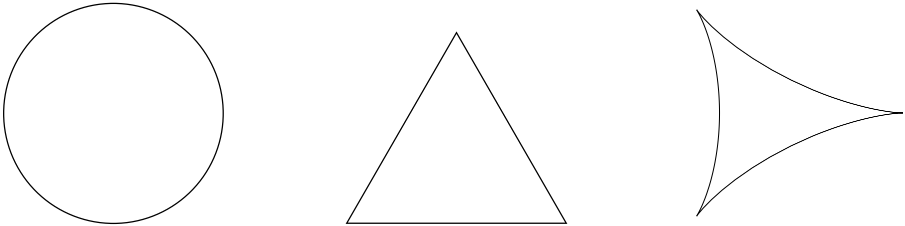

Question:
How small can such sets B and K be?

History - Classical results

Theorems

- There exists a Besicovitch set of measure zero (Besicovitch, 1920).
- For any $\varepsilon > 0$ there is a Kakeya needle set with less than ε measure (follows from Besicovitch, 1920, using Pál joins).

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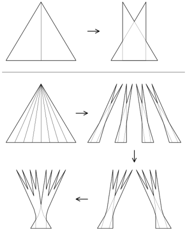

Figure: Perron trees, source: Wikipedia

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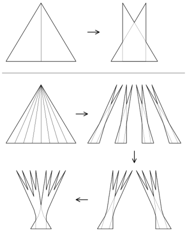

Figure: Perron trees, source: Wikipedia

Remark: A Kakeya needle set must always have positive measure (easy observation).

History - Classical results

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Besicovitch sets in the plane must always have Hausdorff dimension 2 (Davies, 1971).

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Kakeya conjecture

If $B \subset \mathbb{R}^n$ is a Besicovitch set then $\dim B = n$.

A lot of important partial results, but still open for all $n \geq 3$.

History - Classical results

Further results for Kakeya needle sets

- For convex Kakeya needle sets: the smallest possible measure is $1/\sqrt{3}$, that of the equilateral triangle of height 1 (Pál, 1920).

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Cunningham asked, what happens for circular arcs in place of line segments?

Observation

The following are equivalent:

- For any $\varepsilon > 0$ there exists a Kakeya needle set of measure less than ε.
- For any $\varepsilon > 0$ and any two unit line segments in the plane, the first segment can be continuously moved to the second segment within a set of less than ε measure.

History - curved Kakeya needle sets

Definition

$A \subset \mathbb{R}^2$ has the strong Kakeya property (K^s), if for any $\varepsilon > 0$, A can be continuously moved to any prescribed position within a set of less than ε measure.

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- If A has $\dim_H A < 1$, then A has property K^s (Easy observation, H-Laczkovich, 2016).

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- If A has $\dim_H A < 1$, then A has property K^s (Easy observation, H-Laczkovich, 2016).
- If A is a full circle then A does not have property K^s (Easy observation, H-Laczkovich, 2016).

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- If A is a circular arc less than a half circle, then A has K^s (Chang-Csörnyei, 2020).
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  In fact, the same is true if we adjoin the opposite circular arc to A.
- There can be no curves with property (K^s) other than line segments and circular arcs (Csörnyei-H-Laczkovich, 2017).
  More precisely, if A is a closed set with property (K^s) such that all of its connected components are non-trivial, then A must be a subset of parallel line segments or concentric circles.

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Remark: it is an open question whether circular arcs that are at least a half circle and less than a full circle have property (K^s).

Besicovitch sets of curves

Let $\Gamma \subset \mathbb{R}^2$ with $\dim \Gamma = 1$, where $\dim$ denotes Hausdorff dimension.

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$E \subset \mathbb{R}^2$ is a Γ-Besicovitch set if E contains a rotated (and translated) copy of Γ in every direction.

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What can we say about other curves Γ?

Besicovitch sets of curves

Let $\Gamma \subset \mathbb{R}^2$ with $\dim \Gamma = 1$.
Let $\mathcal{H}^1$ denote the 1-dimensional Hausdorff measure.

Definition

We say that $E \subset \mathbb{R}^2$ is an almost-Γ-Besicovitch set if E contains a rotated (and translated) copy of a full-measure subset of Γ in almost every direction.

That is: for almost every direction θ, there exists $\Gamma_\theta \subset \Gamma$ with $\mathcal{H}^1(\Gamma_\theta) = \mathcal{H}^1(\Gamma)$ such that E contains a θ-rotated (and translated) copy of Γ_θ.

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Recall the definition of rectifiable sets: $\Gamma \subset \mathbb{R}^2$ is 1-rectifiable if there are Lipschitz maps $f_i : \mathbb{R} \to \mathbb{R}^2 (i = 1, 2, \dots)$ such that $\mathcal{H}^1(\Gamma \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R})) = 0$.

Theorem (Chang-Csörnyei, 2020)

If Γ is a 1-rectifiable set, then there exists an almost-Γ-Besicovitch set E with measure zero.

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Remarks:

- If Γ is the graph of a convex function, then Γ_θ can be chosen to be Γ minus a single point.
- In fact, the position (rotation and translation) can be chosen continuously, so we have a continuous movement, but $\theta \to \Gamma_\theta$ can not be continuous.
- The directions of tangents play an important role in the proof.

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Question: What can we say about the Hausdorff dimension of (almost-)Γ-Besicovitch sets for rectifiable curves Γ?

Remark: One naturally expects that for dimension, the answer for Γ-Besicovitch sets is the same as for almost-Γ-Besicovitch sets.

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- If Γ is a C^∞ curve other than a circular arc, then every Γ-Besicovitch set E must have Hausdorff dimension 2 (follows from Zahl's paper (2013), unpublished observation of Chang).

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Proof idea for the latter:
If Γ is not a line segment or a circular arc, then there is a portion Γ' of the curve where the curvature is strictly monotone. For such curves, the "cinematic curvature condition" holds, which roughly means that two copies of Γ' can not be tangent to second order, and then Zahl's theorem can be applied to get $\dim E = 2$.

Besicovitch sets of Cantor graphs

Question

What can we say about the dimension of Γ-Besicovitch sets for curves Γ without nice differentiability properties? For example, for rectifiable curves with a vertical tangent $\mathcal{H}^1$-almost everywhere?

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We consider Cantor-graphs (Devil's staircases), only above the Cantor set (this means, that the usual horizontal line segments are not included) with a vertical tangent at almost every point.

The simplest case is: the usual Devil's staircase that maps the $1/3$ Cantor set to $[0, 1]$.

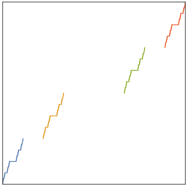

Figure: Devil's staircase (without horizontal line segments)

Besicovitch sets of Cantor graphs

Our result in this simplest case is the following.

Theorem (Altaf-Csörnyei-H, 2022)

Let C be the standard ternary Cantor set and let $s = \dim C_H = \log 2/ \log 3$. Let Γ be the Devil's staircase graph built on C. Then for any Γ-Besicovitch set E, $\dim E \geq 1/s$.

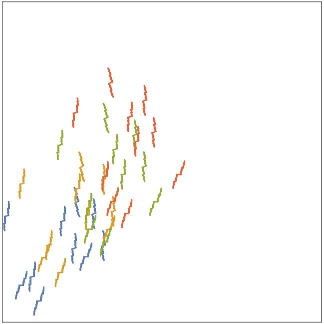

Figure: A portion of a Γ-Besicovitch set E for the Devil's staircase Γ

Besicovitch sets of Cantor graphs

More generally, we consider digit-based self-similar Cantor sets as follows. Let $a, b \in \mathbb{N}$ with $2 \le b < a$, fix a digit set $J \subset \{0, 1, \ldots, a-1\}$ with $|J| = b$, and fix a bijection $\sigma : J \to \{0, 1, \ldots, b-1\}$.
Define

$$C = \left\{ \sum_{j=1}^{\infty} \frac{x_j}{a^j} : x_j \in J \right\}, \ \Gamma = \left\{ \left(\sum_{j=1}^{\infty} \frac{x_j}{a^j}, \sum_{j=1}^{\infty} \frac{\sigma(x_j)}{b^j} \right) : x_j \in J \right\}.$$

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Then the following are easy to see:

- C is self-similar and $\dim C = \log b / \log a$.
- Γ is a rectifiable set with vertical tangents $\mathcal{H}^1$-almost everywhere.

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Theorem (Altaf-Csörnyei-H, 2022)

For any Γ-Besicovitch set E, we have $\dim E \ge \min(2 - s^2, 1/s)$, where $s = \dim C$.

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Remark: in our paper, the results are in fact more flexible, we consider Cantor sets C that are not necessarily self-similar but still have a rigid structure, and more general Cantor graphs on C again with a rigid structure.

Besicovitch sets of Cantor graphs

Recall C and Γ as before:

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Theorem (Altaf, 2023)

For any $s \in [0, 1]$, there exists a Cantor set (a compact nowhere dense perfect set) C with $\dim_H C = s$, a monotone function over C such that its graph Γ is rectifiable with vertical tangents $\mathcal{H}^1$-almost everywhere, and such that there exists a Γ-Besicovitch set E with $\dim_H E = 1$.

Besicovitch sets of Cantor graphs - proof idea

Let E be a Γ-Besicovitch set where Γ is as above, and let $d = \min(2 - s^2, 1/s)$.

Besicovitch sets of Cantor graphs - proof idea

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Fix a small scale $\delta = 1/a^n$. Recall that $s = \dim C = \log b / \log a$, thus we have $\delta^s = 1/b^n$.

We discretize the setting at scale δ:

- Cover Γ by δ^{-s} many rectangles of size $\delta \times \delta^s$. Let R denote the union of these rectangles.
- Chose a maximal δ-separated set of angles A. We have $|A| \approx \delta^{-1}$.
- For each $\theta \in A$, consider the rotated copy $\Gamma_\theta \subset E$ and the corresponding approximation R_θ.

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We will show that

$$\mathcal{L}\left(E(\delta)\right) \approx \mathcal{L}\left(\bigcup_{\theta \in A} \Gamma_\theta(\delta)\right) \gtrsim \frac{\delta^{2-d}}{\log(1/\delta)},$$

where $E(\delta)$ denotes the δ-neighborhood of E, and $\mathcal{L}$ denotes the Lebesgue measure.

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This implies that the Minkowski dimension of E is at least d. Using a standard pigeonholing argument, we also get that $\dim E \geq d$.

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We use the Cauchy-Schwarz inequality:

$$\sum_{\theta \in A} \mathcal{L}(R_\theta) \leq \mathcal{L}\left(\bigcup_{\theta \in A} R_\theta\right)^{1/2} \cdot \left(\sum_{\theta, \theta' \in A} \mathcal{L}(R_{\theta'} \cap R_\theta)\right)^{1/2}.$$

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For the left hand side, we have:

$$\sum_{\theta \in A} \mathcal{L}(R_\theta) = \sum_{\theta \in A} \delta \cdot \delta^s \cdot \delta^{-s} = 1.$$

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Otherwise $|\theta - \theta'| \geq \delta$. WLOG we can assume that one of the graphs is unmoved, that is, $\theta' = 0$, $R_{\theta'} = R$. Assume for simplicity that $\theta > 0$.

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- For each $\theta \in A$, rectangle T of R and T' of R_θ, we have $\mathcal{L}(T \cap T') \lesssim \frac{\delta^2}{\theta}$.

  Reason: This follows from the fact that the same estimate holds for two infinite strips of width δ with angle θ.

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- By the previous observations, we have

$$\sum_{\theta \in A, \theta \geq \delta} \mathcal{L}(R_\theta \cap R) \lesssim \sum_{\theta \in A, \theta \geq \delta} L(\delta, \theta) \cdot \frac{\delta^2}{\theta} \lesssim \delta^{1-s} \log\left(\frac{1}{\delta}\right).$$

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- Summing up for $\theta'\in A$, this yields

$$\sum_{\theta,\theta'\in A}\mathcal{L}(R_\theta\cap R_{\theta'})\lesssim\delta^{-s}\cdot\log\left(\frac{1}{\delta}\right).$$

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Recalling the Cauchy-Schartz inequality:

$$1=\sum_{\theta\in A}\mathcal{L}(R_\theta)\leq\mathcal{L}\left(\bigcup_{\theta\in A}R_\theta\right)^{1/2}\cdot\left(\sum_{\theta,\theta'\in A}\mathcal{L}(R_{\theta'}\cap R_\theta)\right)^{1/2},$$

we obtain $\mathcal{L}\left(\bigcup_{\theta\in A}R_\theta\right)\gtrsim\delta^s/\log\left(\frac{1}{\delta}\right)$.

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we obtain $\mathcal{L}\left(\bigcup_{\theta \in A} R_\theta\right) \gtrsim \delta^s / \log\left(\frac{1}{\delta}\right)$.
This gives us $\dim E \geq 2 - s$.

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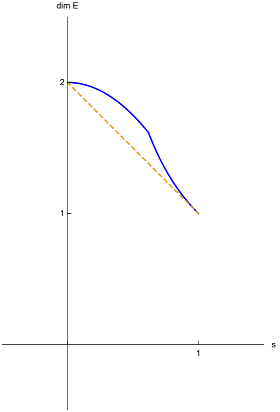

Figure: The bounds $2 - s$ and $\min(2 - s^2, 1/s)$

In order to improve $\dim E \geq 2 - s$ to $\min(2 - s^2, 1/s)$, we give an improved bound for $L(\delta, \theta) \lesssim \delta^{-s}$.

Besicovitch sets of Cantor graphs

The improvement on $L(\delta, \theta)$ is done separately for small angles and large angles. We say that θ is small if $\theta \leq \min(\delta^s, \delta^{1-s})$, otherwise it is large.

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We say that θ is small if $\theta \le \min(\delta^s, \delta^{1-s})$, otherwise it is large.
For small angles, we use the rigid structure of C and Γ. We obtain

$$L(\delta, \theta) \lesssim \frac{1}{\delta^s} \cdot \max\left(\frac{\delta}{\theta}, \theta^s\right).$$

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For large angles, we use 3 different scales: $r > \rho > \delta$ carefully chosen to satisfy the following properties:

- θ is considered to be small with respect to scale r. More precisely, $\theta \approx \min(r^s, r^{1-s})$.
- Inside an intersecting pair of $\rho \times \rho^s$ rectangles, only constant many pairs of $\delta \times \delta^s$ rectangles can intersect. For this, we need $\rho \approx \theta \cdot \delta^s$.

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We get the following bound:

$$L(\delta, \theta) \lesssim L(\rho, \theta) \lesssim \frac{r^s}{\rho^s} \cdot L(r, \theta) \lesssim \frac{r^s}{(\theta\delta^s)^s} \cdot L(r, \theta).$$

Finally, we use the small angles bound for $L(r, \theta)$.

Thank you for your attention!