Some Algebraic Tools in Fractal Geometry and in Metric Number Theory

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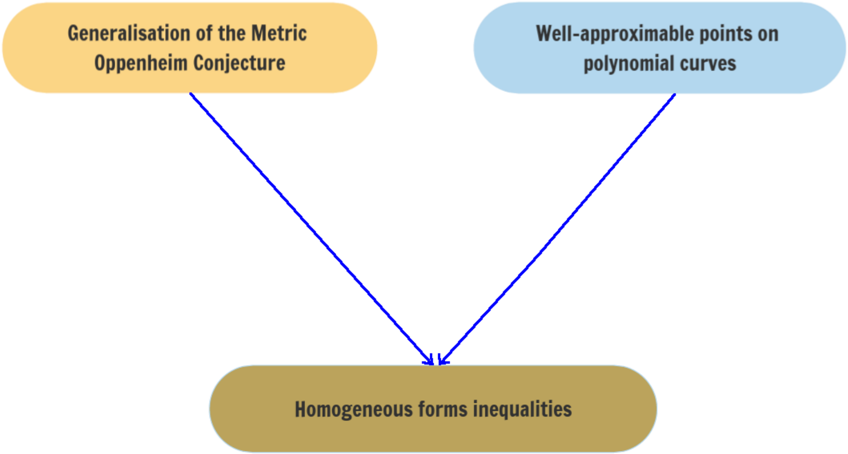
Joint work with...

Oscar Marmon (Lund University, Sweden)

Generalisation of the Metric Oppenheim Conjecture

Well-approximable points on polynomial curves

Homogeneous forms inequalities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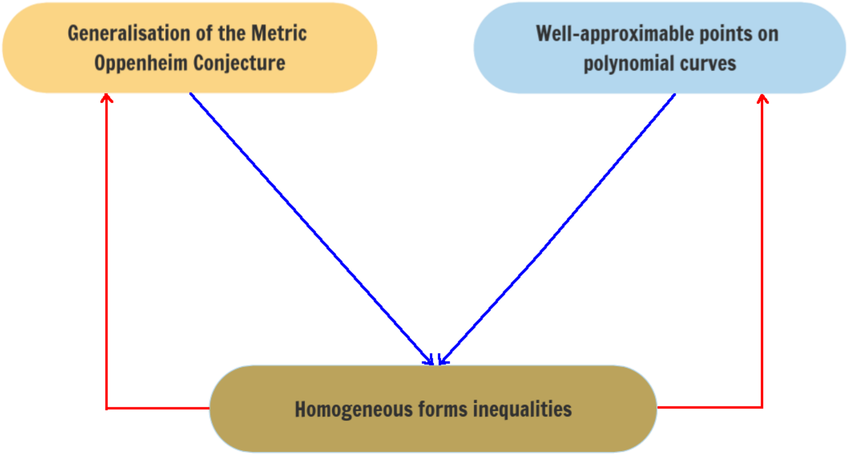

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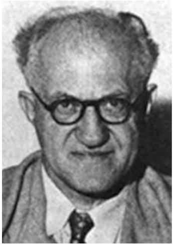
The Oppenheim Conjecture

Theorem (Oppenheim's conjecture, 1929)

Let $Q(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$ be a nondegenerate indefinite quadratic form in $n \geq 3$ variables which is not a real multiple of a rational form. Then for any $\varepsilon > 0$, there exists $\boldsymbol{m} \in \mathbb{Z}^n \backslash \{\boldsymbol{0}\}$ such that $0 < |Q(\boldsymbol{m})| < \varepsilon$.

- This strong form is actually due to Davenport (1946).



Alexander Oppenheim (1903–1997) & Harold Davenport (1907–1969)

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- The assumption that $n \geq 3$ is crucial :

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- The assumption that $n \geq 3$ is crucial : if $\alpha \in$ *Bad* and if $Q(x_1, x_2) = x_1 \cdot (x_1\alpha - x_2)$, then, by definition, there exists $c > 0$ such that

$$\forall (p, q) \in \mathbb{Z}^2 \backslash \{\boldsymbol{0}\}, \qquad |Q(p, q)| \; = \; |q \cdot (q\alpha - p)| > c.$$

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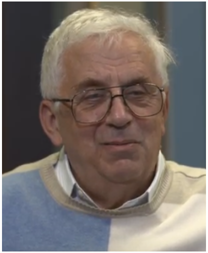
- The conjecture was proved by Margulis in 1987.



Gregori Aleksandrovitch Margulis (1946 –)

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- Quantitative versions :

Question 1 : given $\varepsilon > 0$, determine

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Question 2 : given $\varepsilon > 0$, determine the asymptotic behavior (as $T \to \infty$) of the counting function

$$\mathcal{N}_Q(\varepsilon, T) = \# \{ \boldsymbol{m} \in \mathbb{Z}^n \backslash \{\boldsymbol{0}\} \; : \; \|\boldsymbol{m}\|_2 \leq T \text{ and } |Q(\boldsymbol{m})| < \varepsilon \} .$$

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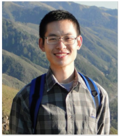
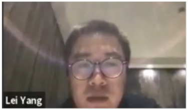
Theorem (Lindenstrauss, Mohammadi, Wang & Yang — 2023)

There exist an absolute constant $\kappa > 0$ such that if none of the coefficients of $Q(\boldsymbol{x})$ is a Liouville number, then

$$M_Q(\varepsilon) \ll_Q \varepsilon^{-\kappa}.$$



Elon Lindenstrauss, Amir Mohammadi, Zhiren Wang & Lei Yang

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Theorem (Eskin, Margulis, Mozes — 1998)

If Q has signature $p \geq 3$ and $q \geq 1$, then

$$\mathcal{N}_Q(\varepsilon, T) \underset{T\to\infty}{\sim} Vol_n\left(\{\boldsymbol{x} \in B(\boldsymbol{0}, T) : |Q(\boldsymbol{x})| < \varepsilon\}\right) \underset{T\to\infty}{\sim} c_Q \cdot \varepsilon \cdot T^{n-2}$$

Alex Eskin, Grigori Margulis & Shahar Mozes

The Oppenheim Conjecture : generalisation

Problem: Can similar (quantitative) results be obtained in the case of *any* homogeneous form $F(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$ of degree $d \geq 3$?

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Problem (Metric approach, Athreya & Margulis — 2018)

Do the quantitative forms of the Oppenheim conjecture hold for ***almost all*** *unimodular distorsions of the homogeneous form $F(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$?*

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for **almost all** $\mathfrak{g} \in SL_n(\mathbb{R})$.

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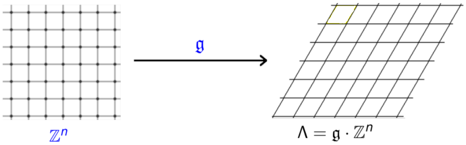
- The map $\boldsymbol{m} \in \mathbb{Z}^n \mapsto \mathfrak{g} \cdot \boldsymbol{m} \in \mathfrak{g} \cdot \mathbb{Z}^n$ sends the integer lattice $\mathbb{Z}^n$ onto the unimodular lattice $\Lambda = \mathfrak{g} \cdot \mathbb{Z}^n$.



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$\rightarrow$ One is considering **quantitative versions** of the generalised Oppenheim conjecture when the solutions are **randomly** drawn from a unimodular lattice.

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- This quotient can be equipped with a natural (Haar) **probability** measure μ_n (descending from $SL_n(\mathbb{R})$).

Recall : $X_n = SL_n(\mathbb{R})/SL_n(\mathbb{Z})$.

Theorem (Siegel — 1945)

For any $f \in \mathbb{L}^1(\mathbb{R}^n)$,

$$\int_{X_n}\left(\sum_{\lambda\in\Lambda} f(\lambda)\right)\cdot d\mu_n(\Lambda) = \int_{\mathbb{R}^n} f(\mathbf{x})\cdot d\mathbf{x}.$$

In particular, if $A\subset\mathbb{R}^n$ *is measurable and if* $f=\chi_A$*, then* $\mu_n(\Lambda\in X_n : \Lambda\cap A\neq\emptyset)\leq |A|$.

Carl Siegel (1896–1981)

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- Working slightly harder with (essentially) $f = \chi_{A^c}$:

Theorem (Athreya & Margulis — 2009)

If $A \subset \mathbb{R}^n$ has positive measure, then

$$\mu_n(\Lambda \in X_n : \Lambda \cap A = \emptyset) \ll_n |A|^{-1}.$$

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$$M_F(\varepsilon, \mathfrak{g}) \ = \ \min\left\{\|\boldsymbol{m}\|_2 \ : \ 0 < |F(\mathfrak{g}\cdot\boldsymbol{m})| < \varepsilon\right\}.$$

Step 1 : Let $\varepsilon = 2^{-j}$, $h \ : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing function and

$$A(j) \ := \ \left\{\boldsymbol{x} \in \mathbb{R}^n \ : \ \|\boldsymbol{x}\| \le h\left(2^j\right) \text{ and } 0 < |F(\boldsymbol{x})| < 2^{-j}\right\}.$$

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Step 2 : From the Borel-Cantelli lemma, if

$$\sum_{j=1}^{\infty} \mu_n \left(\Lambda \in X_n \; : \; \Lambda \cap A(j) = \emptyset\right) \; < \; \infty,$$

then $M_F\left((2^{-j}, \mathfrak{g}\right) \ll_{\mathfrak{g}} h\left(2^j\right)$ for **almost all** $\mathfrak{g} \in SL_n(\mathbb{R})$;

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Step 3 : From the Athreya-Margulis's Theorem, if

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Conclusion : The problem boils down to estimating the *volume* of the set

$$\mathcal{S}_F(a, b) \ = \ \{\boldsymbol{x} \in \mathbb{R}^n \ : \ \|\boldsymbol{x}\|_2 \leq a \quad \text{and} \quad |F(\boldsymbol{x})| < b\}$$

for suitable values of $a, b > 0$.

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Theorem (Rogers, 1956)

Let $A \subset \mathbb{R}^n$ be measurable and let $M \geq 1$. Then,

$$\mu_n\left(\left\{\Lambda \in X_n \; : \; \left|\#\left(\Lambda \cap A\right) - |A|\right| > M\right\}\right) \; \ll_n \; \frac{|A|}{M^2}.$$

Claude Ambrose Rogers (1920–2005)

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$$\mathcal{N}_F(\varepsilon, T, \mathfrak{g}) \;=\; \#\left\{\boldsymbol{m} \in \mathbb{Z}^n \backslash \{\boldsymbol{0}\} \;:\; \|\boldsymbol{m}\|_2 \leq T \text{ and } |(F \circ \mathfrak{g})(\boldsymbol{m})| < \varepsilon\right\}.$$

Theorem (Rogers, 1956)

Let $A \subset \mathbb{R}^n$ be measurable and let $M \geq 1$. Then,

$$\mu_n\left(\{\Lambda \in X_n \;:\; |\#(\Lambda \cap A) - |A|| > M\}\right) \ll_n \frac{|A|}{M^2}.$$

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Step 2 : From the Borel-Cantelli lemma and from Rogers' Theorem, $\mathcal{N}_F(\varepsilon, T, \mathfrak{g}) \sim_{T\to\infty} c_{\mathfrak{g}} \cdot \left|A_F^{(\varepsilon)}(T)\right|$ for **almost all** $\mathfrak{g} \in SL_n(\mathbb{R})$ if

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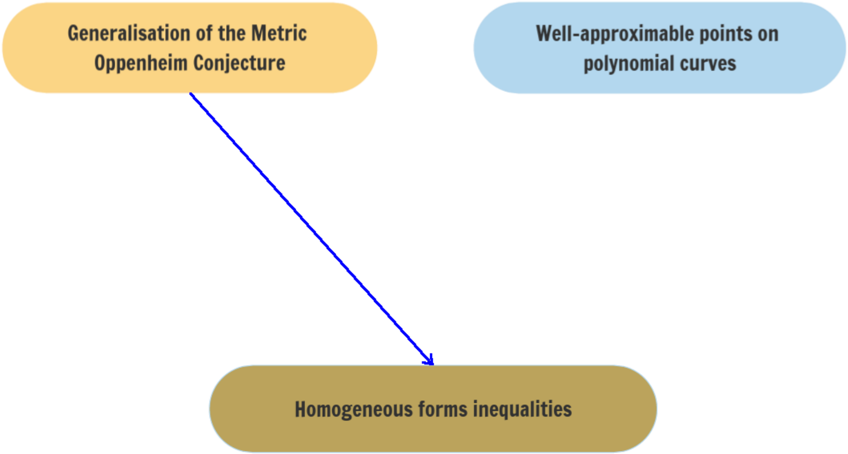
Conclusion : The problem boils down to estimating the *volume* of the set

$$\mathcal{S}_F(a, b) \;=\; \left\{\boldsymbol{x} \in \mathbb{R}^n \;:\; \|\boldsymbol{x}\|_2 \leq a \quad \text{and} \quad |F(\boldsymbol{x})| < b\right\}$$

for suitable values of $a, b > 0$.


Generalisation of the Metric Oppenheim Conjecture
Well-approximable points on polynomial curves
Homogeneous forms inequalities

Summary of the goal

- Given $a, b > 0$, let

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- Taking $a = T$ and $b = b(T)$ a function of a, define

$$\mathcal{S}_F(T, b(T)) = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_2 \leq T \quad \text{and} \quad |F(\boldsymbol{x})| < b(T)\}.$$

Goal : To determine the volume of $\mathcal{S}_F(T, b(T))$ as $T \to \infty$.

- This will solve the problem of generalising the Oppenheim Conjecture from a metric standpoint.

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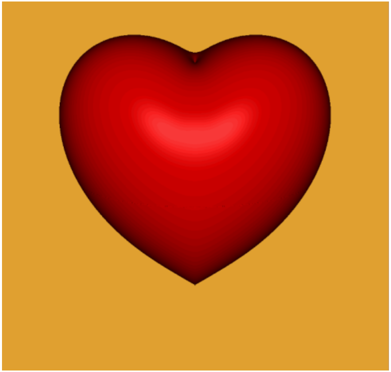
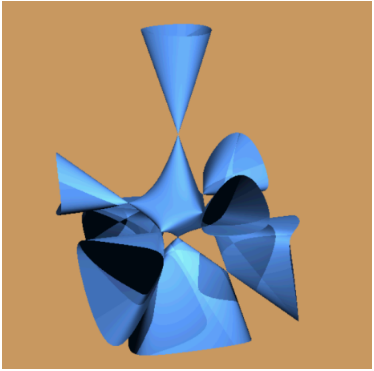
Some semialgebraic sets



$(2x^2 + y^2 + z^2 - 1)^3 - \frac{x^2 z^3}{10} - y^2 z^3 = 0$



The Hessian of the Cayley cubic

Volume estimate

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Let $b_F(-s) \in \mathbb{Q}[s]$ be the Sato-Bernstein polynomial associated to the homogeneous form $F(\boldsymbol{x}) \in \mathbb{R}[\boldsymbol{x}]$. It admits a root $r_F \in \mathbb{Q} \cap (0, n/d]$ with multiplicity $m_F \geq 1$ such that

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On the Sato-Bernstein theory

Theorem (Bernstein – 1971, Satō – 1972)

There exists a nonzero polynomial $B(s) \in \mathbb{R}[s]$ and a differential operator $\mathcal{D}(\boldsymbol{x}, s, \boldsymbol{\partial}) \in \mathbb{R}\langle \boldsymbol{x}, s, \boldsymbol{\partial} \rangle$ such that

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- The Sato-Bernstein polynomial $b_F(s) \in \mathbb{Q}[s]$ of $F(\boldsymbol{x}) \in \mathbb{R}[x]$ is the smallest degree monic polynomial satisfying such a relation.

On the Sato-Bernstein theory : examples

Recall : $b_F(s) \cdot F(\boldsymbol{x})^s = \mathcal{D}(\boldsymbol{x}, s, \partial) F(\boldsymbol{x})^{s+1}$.

- Example 1 : when $F_1(\boldsymbol{x}) = \|\boldsymbol{x}\|^2 = \sum_{i=1}^n x_i^2$,

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In particular, the poles of ζ_F are roots of $b_F(s)$.

On the Sato-Bernstein theory : sketch of proof

Recall : $\langle \zeta_F(s), \psi \rangle = \int_{\mathbb{R}^n} \psi(\boldsymbol{x}) \cdot F(\boldsymbol{x})^s \cdot \mathrm{d}\boldsymbol{x}.$

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Step 2 : *Tauberian Theorem* :

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Step 2 : *Tauberian Theorem* : if ρ_F is the opposite of the largest pole of ζ_F, with order $\mu_F \geq 1$, then the volume estimate holds with (ρ_F, μ_F) in place of (r_F, m_F).

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Step 3 : To show that $(\rho_F, \mu_F) = (r_F, m_F)$

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For $r_F \in \mathbb{Q} \cap (0, n/d]$ root of $b_F(-s) \in \mathbb{Q}[s]$ with multiplicity m_F ,

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- If $F(\boldsymbol{x}_0) = 0$ and $\nabla F(\boldsymbol{x}_0) = \boldsymbol{0}$, then there exists an analytic change of variables φ in a suitable neighbourhood of $\boldsymbol{x}_0$ such that $F(\varphi(u_1, \ldots, u_n)) = \prod_{i=1}^n u_i^{k_i}$.

Back to Question 1'

Question 1' : given $\varepsilon > 0$, determine, for **almost all** $\mathfrak{g} \in SL_n(\mathbb{R})$,

$$M_F(\varepsilon, \mathfrak{g}) \; = \; \min \left\{ \|\boldsymbol{m}\|_2 \; : \; 0 < |F(\mathfrak{g} \cdot \boldsymbol{m})| < \varepsilon \right\}.$$

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Let $h : \mathbb{R}_+ \to (1, \infty)$ be an increasing function such that

$$\limsup_{j \to +\infty} \frac{h\left(2^{j+1}\right)}{h\left(2^j\right)} < \infty \quad \textit{and} \quad \frac{2^j}{h\left(2^j\right)^d} \underset{j \to \infty}{\longrightarrow} 0.$$

If

$$\sum_{j=0}^{\infty} \frac{2^{j r_F}}{h\left(2^j\right)^{n - r_F d} \cdot \left|\log\left(2^j \cdot h\left(2^j\right)^{-d}\right)\right|^{m_F - 1}} < \infty,$$

then, for almost every $\mathfrak{g} \in SL_n(\mathbb{R})$, $M_F(\varepsilon, \mathfrak{g}) \ll_{\mathfrak{g}} h\left(\varepsilon^{-1}\right)$.

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- One can take $h(x) = x^\alpha$ for any $\alpha > (r_F + 1)/(n - r_F d)$ when $r_F < n/d$.

Back to Question 2'

Question 2' : given $\varepsilon > 0$, determine, for **almost all** $\mathfrak{g} \in SL_n(\mathbb{R})$, the asymptotic behavior (as $T \to \infty$) of the counting function

$$\mathcal{N}_F(\varepsilon, T, \mathfrak{g}) \;=\; \#\left\{\boldsymbol{m} \in \mathbb{Z}^n \backslash \{\boldsymbol{0}\} \;:\; \|\boldsymbol{m}\|_2 \leq T \text{ and } \left|(F \circ \mathfrak{g})(\boldsymbol{m})\right| < \varepsilon\right\}$$

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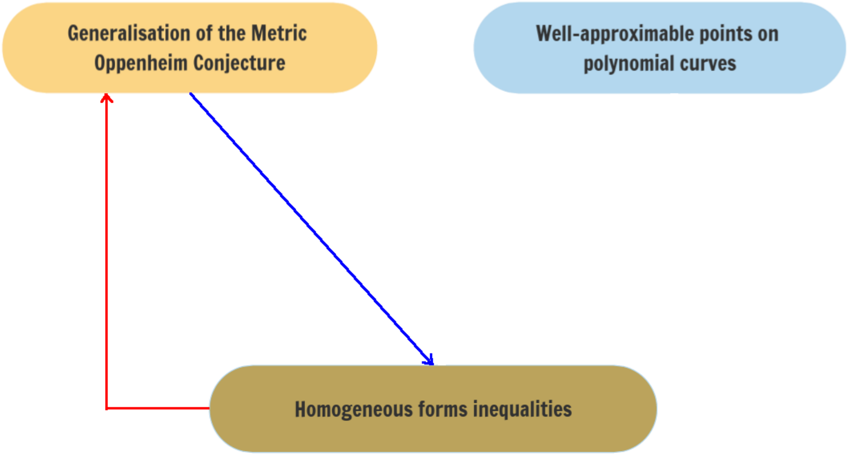
Theorem (A. & Marmon, 2023+)

If $r_F < n/d$, then for almost every $\mathfrak{g} \in SL_n(\mathbb{R})$,

$$\mathcal{N}_F(\varepsilon, T, \mathfrak{g}) \underset{T \to \infty}{\sim} c_F(\mathfrak{g}) \cdot T^{n - r_F d} \cdot (\log T)^{m_F - 1} \cdot \varepsilon^r.$$


Generalisation of the Metric Oppenheim Conjecture
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Approximation on polynomial curves

Problem

Given $\tau > 0$, determine the Hausdorff dimension of the set

$$W_3(\tau) = \left\{ x \in \mathbb{R} \ : \ \left| x - \frac{p}{q} \right| < \frac{1}{q^\tau} \quad \textit{and} \quad \left| x^3 - \frac{r}{q} \right| < \frac{1}{q^\tau} \quad \textit{i.o.} \right\}.$$

- More generally, fix $P(x) \in \mathbb{R}[x]$ of degree $d \geq 3$ and define

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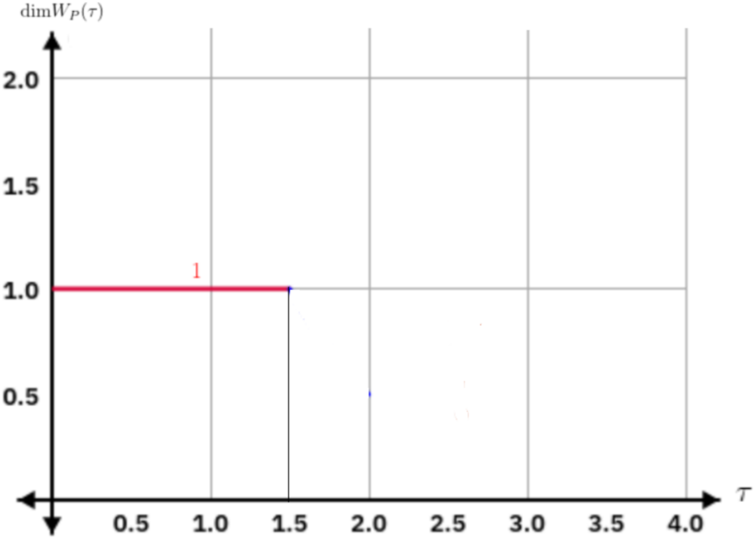
Approximation on polynomial curves : what is known

Recall: $W_P(\tau) = \{x \in \mathbb{R} : \max\{|x - p/q|, |P(x) - r/q|\} < q^{-\tau} \text{ i.o.}\}$

- From Dirichlet's Theorem, for *any* $(x, y) \in \mathbb{R}^2$,

$$\max\left\{\left|x - \frac{p}{q}\right|, \left|y - \frac{r}{q}\right|\right\} < \frac{1}{q^{3/2}} \qquad \text{i.o.}$$

Approximation on polynomial curves : what is known



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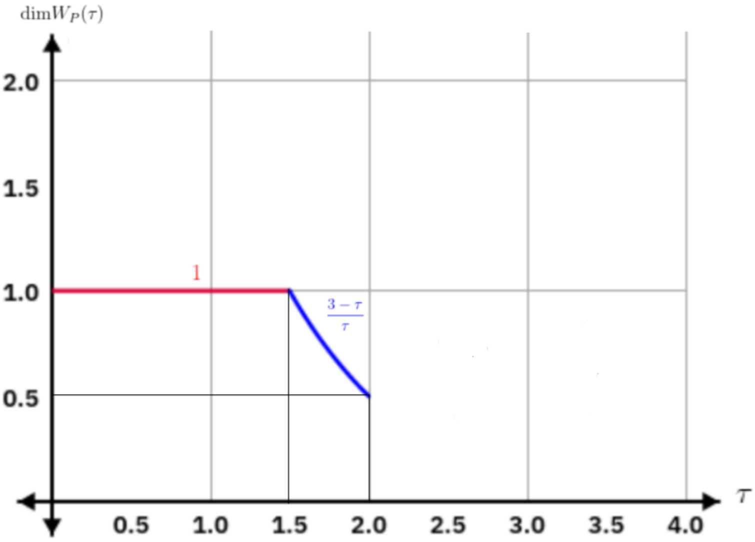
Theorem (Beresnevich, Dickinson, Velani — 2007)

Let I be an interval and let $\tau \in (3/2, 2)$. Assume that $f \in \mathcal{C}^3(I)$ is such that $\dim\{x \in I \ : \ f''(x) = 0\} \leq (3-\tau)/\tau$. Then,

$$\dim W_f(\tau) \ = \ \frac{3-\tau}{\tau}.$$



Victor Beresnevich, Detta Dickinson & Sanju Velani


$\dim W_P(\tau)$
2.0
1.5
1.0
0.5
1
$\frac{3-\tau}{\tau}$
τ
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0

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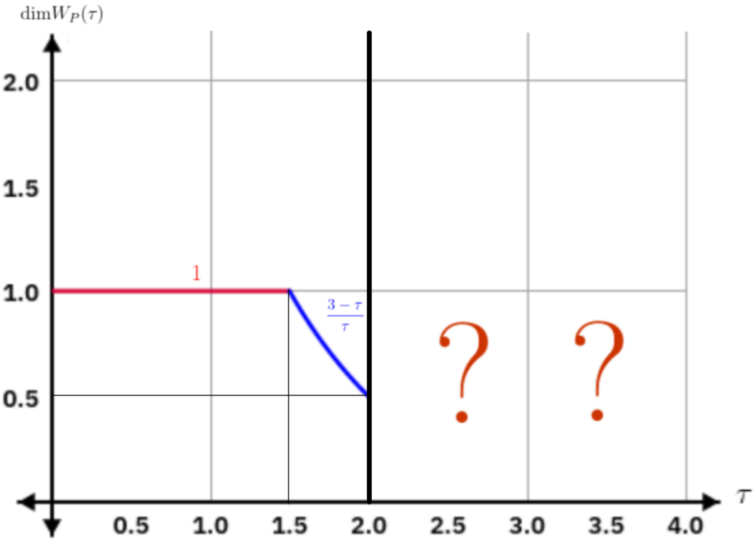

Approximation on polynomial curves : analysis

Recall: $W_P(\tau) = \{x \in \mathbb{R} : \max\{|x - p/q|, |P(x) - r/q|\} < q^{-\tau} \text{ i.o.}\}$

- Assume that $|x - p/q|, |P(x) - r/q| < q^{-\tau}$ for some x in a bounded interval and some $Q/2 \leq q \leq Q$. By a Taylor expansion :

$$P\left(\frac{p}{q}\right) = P\left(x + \left(\frac{p}{q} - x\right)\right)$$
$$= P(x) + O(|x - p/q|) = \frac{r}{q} + O\left(\frac{1}{q^\tau}\right).$$

- Thus, one needs to count the number of integer solutions to the homogeneous form inequality

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Approximation on polynomial curves : synthesis

Question 3 : determine the Hausdorff dimension of the set

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Conclusion : The problem boils down to estimating the *number of integer points* in the set

$$S_F(a,b) = \{ \boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_2 \leq a \quad \text{and} \quad |F(\boldsymbol{x})| < b \}$$

for suitable values of $a, b > 0$ (when $n = 3$).

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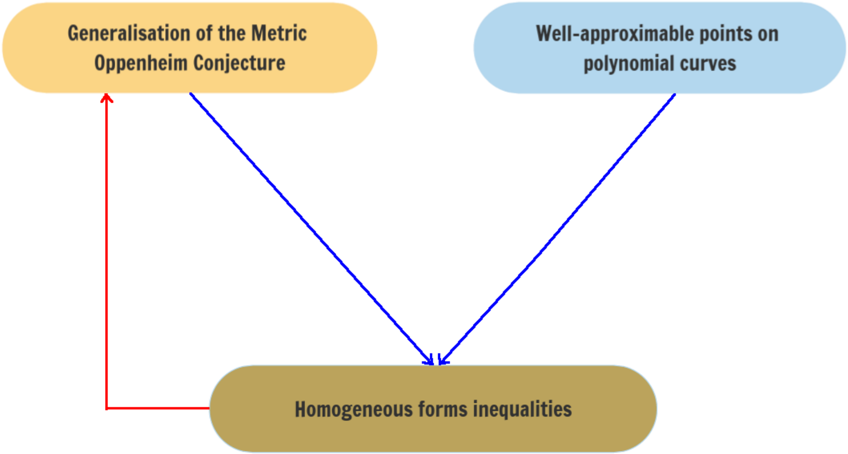
Generalisation of the Metric Oppenheim Conjecture
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Summary of the goal

- Given $a, b > 0$, let

$$\mathcal{S}_F(a,b) = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_2 \leq a \quad \text{and} \quad |F(\boldsymbol{x})| < b\}.$$

- Taking $a = T$ and $b = b(T)$ a function of a, define

$$\mathcal{S}_F(T, b(T)) = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_2 \leq T \quad \text{and} \quad |F(\boldsymbol{x})| < b(T)\}.$$

Goal : To determine $\#\left(\mathbb{Z}^n \cap \mathcal{S}_F(T, b(T))\right)$ as $T \to \infty$.

- This is to tackle the problem of simultaneous approximation on polynomial curves.

Goal 2 : To determine $\#\left(\mathbb{Z}^n \cap \mathcal{S}_F\left(T, b(T)\right)\right)$ as $T \to \infty$, where $\mathcal{S}_F\left(T, b(T)\right) = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_2 \leq T \text{ and } |F(\boldsymbol{x})| < b(T)\}$.

Principle :

$$\#\left(\mathbb{Z}^n \cap \mathcal{S}_F\left(T, b(T)\right)\right) \;=\; \sum_{\boldsymbol{k}\in\mathbb{Z}^n} \chi_{B_2(\boldsymbol{0},1)}\left(\frac{\boldsymbol{k}}{T}\right) \cdot \chi_{[-1,1]}\left(\frac{F(\boldsymbol{k})}{b(T)}\right)$$

Counting lattice points : principle

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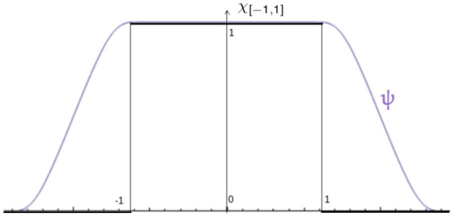
$$\leq \sum_{\boldsymbol{k}\in\mathbb{Z}^n} \xi\left(\frac{\boldsymbol{k}}{T}\right)\cdot \psi\left(\frac{F(\boldsymbol{k})}{b(T)}\right),$$

where ξ and ψ are smooth and compactly supported, bounding from above the characteristic functions.



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&\leq \sum_{\boldsymbol{k}\in\mathbb{Z}^n} \xi\left(\frac{\boldsymbol{k}}{T}\right)\cdot\psi\left(\frac{F(\boldsymbol{k})}{b(T)}\right) \\
&= \sum_{\boldsymbol{k}\in\mathbb{Z}^n} \left(\widehat{\xi\left(\frac{\cdot}{T}\right)\cdot\psi\left(\frac{F(\cdot)}{b(T)}\right)}\right)(\boldsymbol{k})
\end{aligned}$$

from the Poisson Summation formula, where

$$\left(\widehat{\xi\left(\frac{\cdot}{T}\right)\cdot\psi\left(\frac{F(\cdot)}{b(T)}\right)}\right) : \boldsymbol{y}\in\mathbb{R}^n \mapsto \int_{\mathbb{R}^n} \xi\left(\frac{\boldsymbol{u}}{T}\right)\cdot\psi\left(\frac{F(\boldsymbol{u})}{b(T)}\right)\cdot e\left(\boldsymbol{y}\cdot\boldsymbol{u}\right)\cdot d\boldsymbol{u}.$$

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$$\#\left(\mathbb{Z}^n \cap \mathcal{S}_F\left(T, b(T)\right)\right) \leq \sum_{\boldsymbol{k} \in \mathbb{Z}^n} \widehat{\left(\xi\left(\frac{\cdot}{T}\right) \cdot \psi\left(\frac{F(\cdot)}{b(T)}\right)\right)}(\boldsymbol{k})$$

Counting lattice points : principle

Goal 2 : To determine $\#\left(\mathbb{Z}^n \cap \mathcal{S}_F\left(T, b(T)\right)\right)$ as $T \to \infty$, where $\mathcal{S}_F\left(T, b(T)\right) = \left\{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_2 \leq T \text{ and } |F(\boldsymbol{x})| < b(T)\right\}$.

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$$= \left(\xi\left(\frac{\cdot}{T}\right) \cdot \widehat{\psi\left(\frac{F(\cdot)}{b(T)}\right)}\right)(\boldsymbol{0})$$

$$+ \left(\sum_{1\leq\|\boldsymbol{k}\|\leq M} + \sum_{\|\boldsymbol{k}\|>M}\right) \left(\xi\left(\frac{\cdot}{T}\right) \cdot \widehat{\psi\left(\frac{F(\cdot)}{b(T)}\right)}\right)(\boldsymbol{k}).$$

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Counting lattice points : geometric tomography

Key step : to estimate the decay of

$$\int_{\mathbb{R}^n} \xi\left(\frac{\boldsymbol{u}}{T}\right) \cdot \psi\left(\frac{F(\boldsymbol{u})}{b(T)}\right) \cdot e(\boldsymbol{k} \cdot \boldsymbol{u}) \cdot d\boldsymbol{u} \sim \int_{\mathbb{R}^n} \chi_{B_2}\left(\frac{\boldsymbol{u}}{T}\right) \cdot \chi_{[-1,1]}\left(\frac{F(\boldsymbol{u})}{b(T)}\right) \cdot e(\boldsymbol{k} \cdot \boldsymbol{u}) \cdot d\boldsymbol{u}$$

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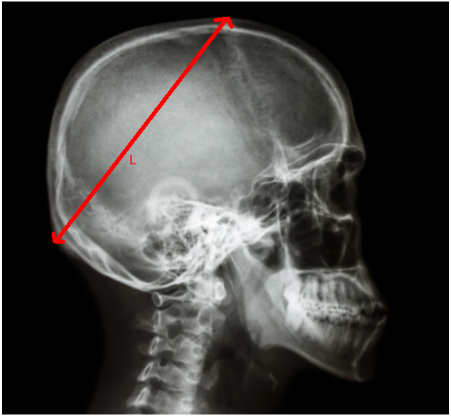

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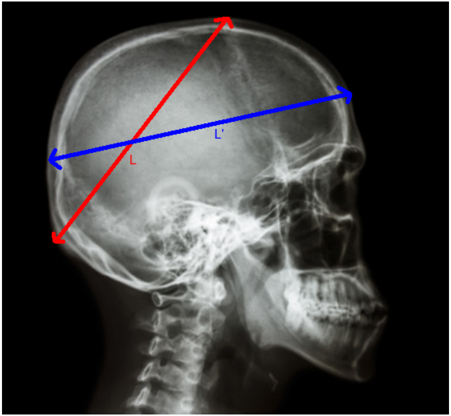

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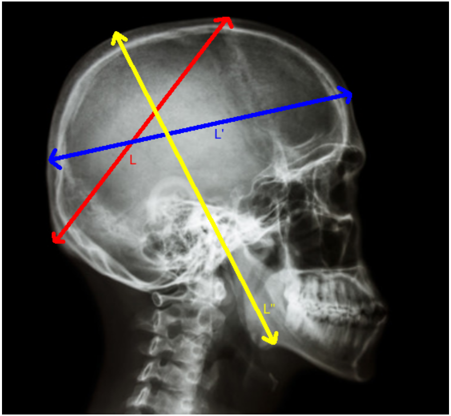

Counting lattice points : geometric tomography

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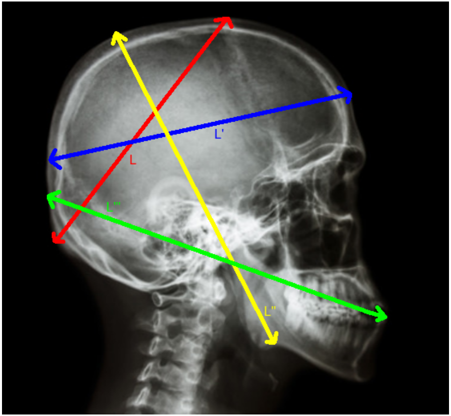

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Formalisation: decompose $\boldsymbol{k} = \lambda\cdot\boldsymbol{v}$ with $\lambda = \|\boldsymbol{k}\| > 0$ and $\boldsymbol{v}\in\mathbb{S}^{n-1}$ and make the *"change of variables"*

$$\boldsymbol{u} = (u_1, \dots, u_n) \mapsto (u_1, \dots, u_{n-1}, \boldsymbol{v}\cdot\boldsymbol{u}) = (u_1, \dots, u_{n-1}, \sigma)$$

Counting lattice points : geometric tomography

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Conclusion : one obtains

$$\int_{\mathbb{R}^n} \xi\left(\frac{\boldsymbol{u}}{T}\right)\cdot\psi\left(\frac{F(\boldsymbol{u})}{b(T)}\right)\cdot e(\boldsymbol{k}\cdot\boldsymbol{u})\cdot d\boldsymbol{u} = \int_{\mathbb{R}} \underbrace{\mu\left(\frac{b(T)}{T^d}, \boldsymbol{v}, \sigma\right)}_{\text{Gel'fand–Leray form}}\cdot e(\lambda\sigma)\cdot d\sigma$$

Counting lattice points : geometric tomography

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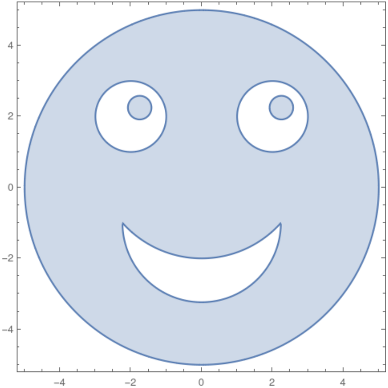

A semialgebraic domain defined by $|F(\boldsymbol{x})| \leq b(T)$ and $\|\boldsymbol{x}\|_2 \leq T$.

Counting lattice points : geometric tomography

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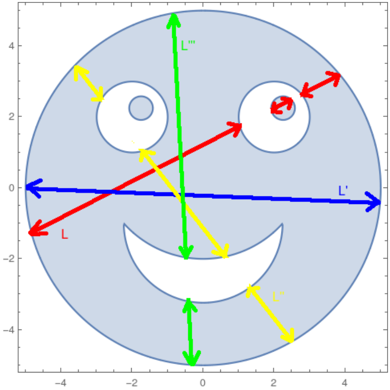

Slices $\boldsymbol{v}\cdot\boldsymbol{x}=\sigma$ of a semialgebraic domain defined by $|F(\boldsymbol{x})|\leq b(T)$ and $\|\boldsymbol{x}\|_2\leq T$.

Counting lattice points : geometric tomography

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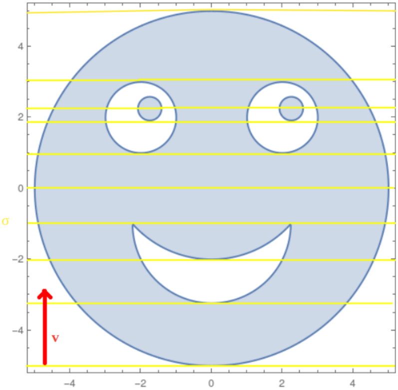

- Properties of the Gel'fand–Leray form :
  - the number of intervals of monotonicity of the map $\sigma \mapsto \mu\left(\varepsilon, \boldsymbol{v}, \sigma\right)$ is bounded above uniformly in the direction $\boldsymbol{v}$.

Counting lattice points : geometric tomography

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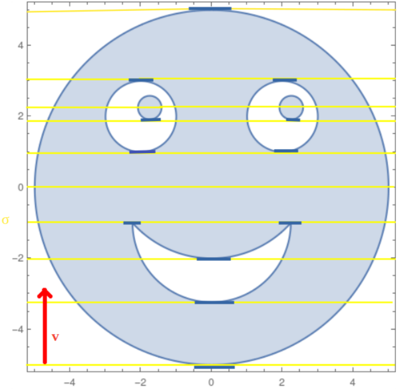

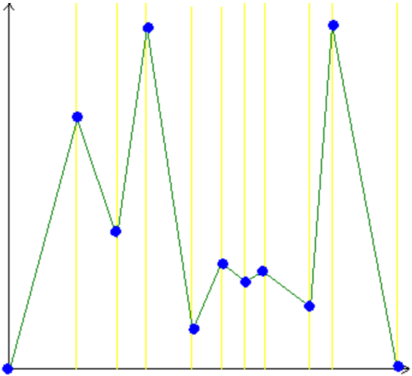

$\sigma \mapsto \mu_T\left(\boldsymbol{v}, \sigma\right)$

Counting lattice points : geometric tomography

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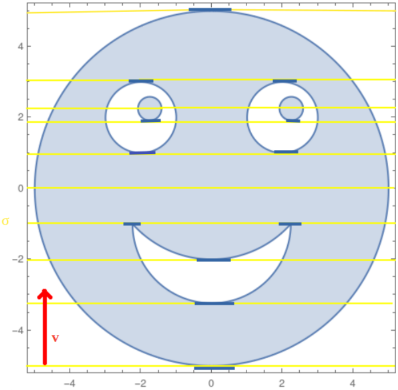

- Properties of the Gel’fand–Leray form : for some $\alpha_F \geq 0$ (a **semialgebraic level of flatness**) and $\beta_F \geq 0$, for $\varepsilon > 0$ small enough,

$$\max_{\substack{\sigma\in\mathbb{R}\\ \boldsymbol{v}\in\mathbb{S}^{n-1}}} \mu(\varepsilon, \boldsymbol{v}, \sigma) \;\asymp\; \varepsilon^{\alpha_F}\cdot|\log\varepsilon|^{\beta_F}\,.$$

- **α_F determines the decay of the Fourier coefficient.**

Counting lattice points : summary

Goal 2 : To determine $\#\left(\mathbb{Z}^n \cap \mathcal{S}_F\left(T, b(T)\right)\right)$ as $T \to \infty$, where $\mathcal{S}_F\left(T, b(T)\right) = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_2 \leq T \text{ and } |F(\boldsymbol{x})| < b(T)\}$.

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where the decay rate of the Fourier coefficients is, **uniformly in $\boldsymbol{k}$**, dictated by the **semialgebraic level of flatness** $\alpha_F > 0$.

Counting lattice points : summary

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Problem (Sarnak — 1997)

Provided that the zero set $F = 0$ is sufficiently curved, there exists $\delta > 0$ such that
$\#\left(\mathbb{Z}^n \cap \mathcal{S}_F\left(T, b(T)\right)\right) \ll Vol_n\left(\mathcal{S}_F\left(T, b(T)\right)\right) + T^{n-1-\delta}$.

Recall : $Vol_n\left(\mathcal{S}_F\left(T, b(T)\right)\right) \asymp T^n \cdot \left(\frac{b(T)}{T^d}\right)^{r_F} \cdot \left|\log\left(\frac{b(T)}{T^d}\right)\right|^{m_F - 1}.$

Peter Sarnak

Counting lattice points : summary

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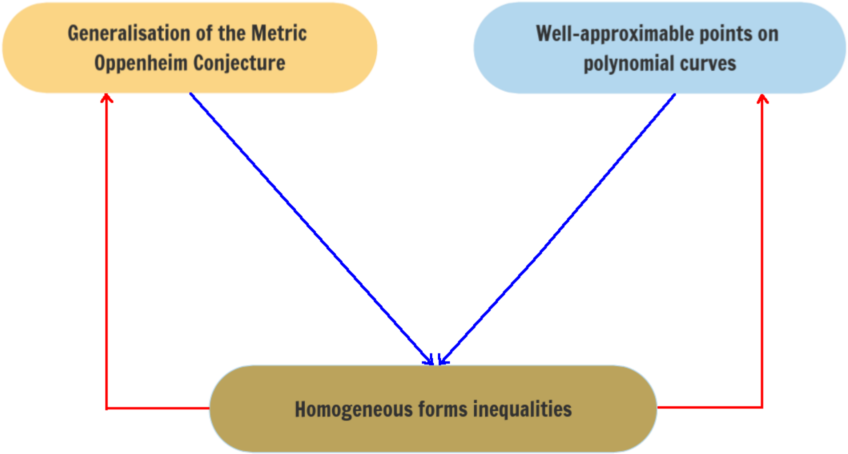
Theorem (A. & Marmon, 2023+ — weak form)

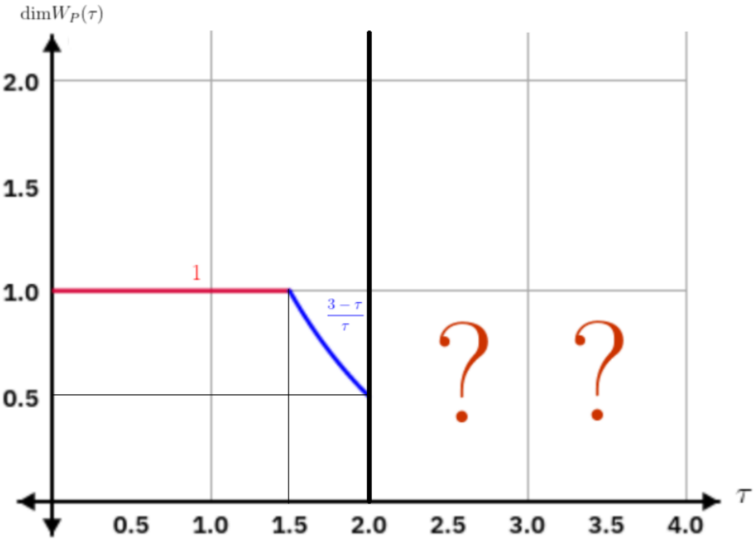
Sarnak's claim holds whenever

$$\alpha_F > \max\left\{r_F - 1, n - 1 - r_F\right\}.$$



Generalisation of the Metric Oppenheim Conjecture
Well-approximable points on polynomial curves
Homogeneous forms inequalities

Approximation on polynomial curves : what is known



Approximation with polynomials : prospective conclusions

Question 3 : when $\tau \geq 2$, determine the Hausdorff dimension of the set

$$W_P(\tau) = \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < \frac{1}{q^\tau} \text{ and } \left| P(x) - \frac{r}{q} \right| < \frac{1}{q^\tau} \text{ i.o.} \right\}.$$

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Expectation : Take $n = 3$ and $b(T) = T^{d-\tau}$.

- Over any compact domain where $P''(x) \neq 0$, one has $r_F = 1$ and $\delta > 1$.

$$\dim W_P(\tau) \begin{cases} = 1 & \text{when } \tau \leq 3/2; \\ = (3-\tau)/\tau & \text{when } 3/2 \leq \tau < 2; \\ = (3-\tau)/\tau & \text{when } 2 \leq \tau < 1 + \delta; \\ \leq 2 - \delta & \text{when } 1 + \delta \leq \tau. \end{cases}$$

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Recall : $Vol_n\left(\mathcal{S}_F(T, b(T))\right) \asymp T^n \cdot \left(\frac{b(T)}{T^d}\right)^{r_F} \cdot \left|\log\left(\frac{b(T)}{T^d}\right)\right|^{m_F - 1}.$

Expectation : Take $n = 3$ and $b(T) = T^{d-\tau}$.

- Over any compact domain where $P''(x) \neq 0$, one has $r_F = 1$ and $\delta > 1$.

$$\dim W_P(\tau) \begin{cases} = 1 & \text{when } \tau \leq 3/2; \\ = (3-\tau)/\tau & \text{when } 3/2 \leq \tau < 2; \\ = (3-\tau)/\tau & \text{when } 2 \leq \tau < 1+\delta; \\ \leq 2 - \delta & \text{when } 1+\delta \leq \tau. \end{cases}$$

Approximation with polynomials : prospective conclusions

Question 3 : when $\tau \geq 2$, determine the Hausdorff dimension of the set

$$W_P(\tau) = \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| < \frac{1}{q^\tau} \text{ and } \left| P(x) - \frac{r}{q} \right| < \frac{1}{q^\tau} \text{ i.o.} \right\}.$$

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To conclude