Families of Markov Maps Associated to Cocompact Fuchsian Triangle Groups

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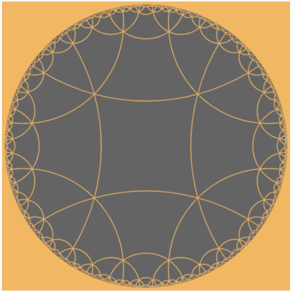
Multifractal Analysis and Self-similarity,
June 2023, CIRM



Motivation

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- Relationship between the Regular Continued Fraction map T and the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{R}$.
- The RCF map T is expansive, Markov, transitive and satisfy Rényi's condition.
- Gauss measure: T-invariant and equivalent to Lebesgue
- Nakada's α-continued fraction maps T_α give a one-parameter deformation of T.

Motivation

Let Γ be a finitely generated discrete subgroup of $SL_2(\mathbb{R})$ acting on $\mathbb{R}$ with dense orbits; i.e. finitely generated Fuchsian group of the first kind.

RCF	B-S
$SL_2(\mathbb{Z})$ acting on $\mathbb{R}$	Γ acting on $\mathbb{D}$
$T:(0,1)\to(0,1)$	f: $\mathbb{S}^1 \to \mathbb{S}^1$
Expansivity, Markov	✓
Transitive, Rényi's condition	✓
Gauss measure	✓
T_α	Analog ?

Rufus Bowen and Caroline Series, *Markov Maps Associated with Fuchsian Groups*, 1979

Motivation

- Katok and Ugarcovici (2017) studied B-S functions associated to cocompact torsion-free Fuchsian groups and defined a multi-parameter deformation family.
- Los (2009) defined Bowen-Series like maps for cocompact surface groups considering the geometric presentation of the group.This study excludes triangle groups.

Main Results

The following is joint work with my Ph.D. advisor, Thomas A. Schmidt.

- Correction to Bowen & Series (1979).

(Q1) Can we define a family of expansive functions via an α-deformation of f in the case of cocompact Fuchsian triangle groups.?

(Q2) If so, is there an ergodic invariant measure for each function in the family?

The Bowen-Series Fundamental Domain

Cocompact Fuchsian Triangle Groups

- A cocompact Fuchsian triangle group Γ is a group with signature $(0; m_1, m_2, m_3)$, where $m_i \in \mathbb{Z}^+$ and $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} < 1$.

Example

(6,6,3) with the presentation

$$\{A, B \mid A^6 = B^6 = (AB)^3 = I\}.$$

- The elements in Γ, as Möbius transformations, act on the unit disc.

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Example

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- The elements in Γ, as Möbius transformations, act on the unit disc.
- The elements in Γ are classified as hyperbolic and elliptic. A hyperbolic element has two fixed points on $\mathbb{S}^1$ and an elliptic element has a single fixed point on the interior of the unit disc.

Constructing a Fundamental Domain

A closed region $\mathscr{F} \subset \mathbb{D}$ is called fundamental domain for Γ if it tesselates $\mathbb{D}$ under the action of Γ.

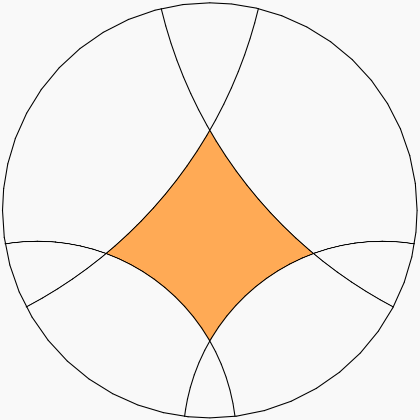


Figure 1: A fundamental domain $\mathscr{F}$ for the triangle group (6,6,3).

Constructing a Fundamental Domain

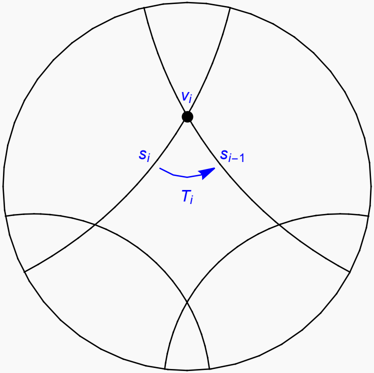


Figure 2: A fundamental domain $\mathscr{F}$ for the triangle group (6,6,3)

Constructing a Fundamental Domain

A fundamental domain $\mathscr{F}$ has extension property if for all $s \in S$,

$$g(s) \cap \bigcup_{T \in \Gamma} T(\mathscr{F}^\circ) = \emptyset,$$

where $g(s)$ represents the geodesic containing the side s.

Constructing a Fundamental Domain

We consider the set $N = \cup_{i=1}^{4} N_i$ in $\mathbb{D}$, where N_i is defined as

$$N_i := \{g \text{ geodesics } \mid v_i \in g \text{ and } \exists T \in \Gamma, T(s_j) \subset g \text{ for some } j\}.$$

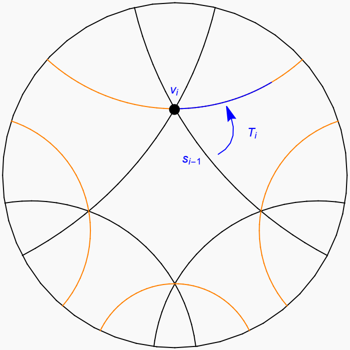


Figure 3: *N* for the group of signature (6,6,3).

Correction to Bowen & Series (1979)

For $\mathscr{F}$ to satisfy the extension property, no geodesic of any N_i meets $\mathscr{F}^\circ$.

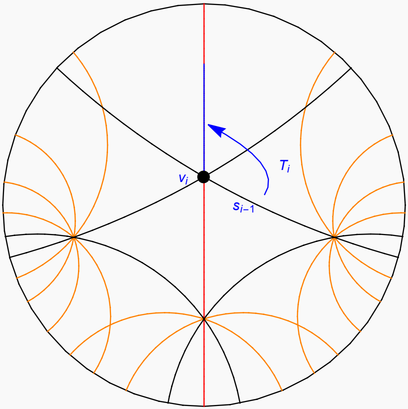


Figure 4: *N* for the group of signature (3,5,6).

Correction to Bowen & Series (1979)

Theorem (Y.K. & Schmidt, 2023)

Suppose that (m_1, m_2, m_3) is the signature of a cocompact hyperbolic Fuchsian triangle group. If more than one m_i is odd, then no convex fundamental domain for the signature has the extension property. Otherwise, the Bowen-Series fundamental domain for this signature does have the extension property.

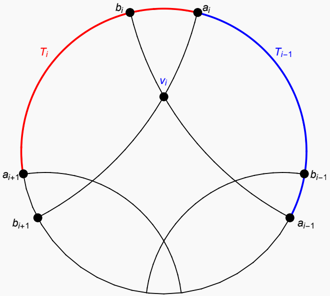
In what follows, we suppose that the Bowen-Series fundamental domain for Γ satisfies extension property.

One-Parameter Deformation of B-S Functions

The Bowen-Series function

The Bowen - Series function f is defined as

$$f(x) := T_i(x) \text{ on } x \in [a_i, a_{i+1}).$$



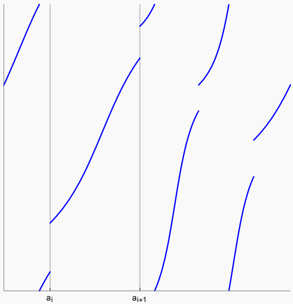


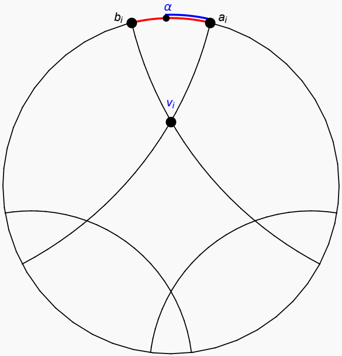
Figure 5: The Bowen-Series function f for the group (6,6,3). Graphed as $y = arg(f(x))$ for $x \in [0, 2\pi)$.

A Function Family

We call $\mathscr{O}_i = [a_i, b_i)$ as the overlap interval and define a function family depending on the parameter α as

$$f_\alpha(x) := \begin{cases} f(x), & x \in \mathbb{S}^1 \setminus \mathscr{D} \\ T_{i-1}(x), & x \in \mathscr{D}, \end{cases}$$

where $\mathscr{D} = [a_i, \alpha)$ is the differing interval.



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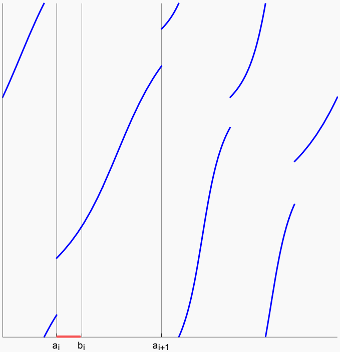

Figure 6: Comparison of the plots of f and f_α

Back to Questions

(Q1) Can we define a family of expansive functions via an α-deformation of f in the case of cocompact Fuchsian triangle groups.? Yes

(Q2) If so, is there an ergodic invariant measure for each function in the family?

Adler-Flatto's 'Folklore Theorem'

Let X be an interval or a circle and $\mathscr{P} = \{I_k\}$ be a finite partition of X. Let $g : X \to X$ satisfy the following:

(1) Piecewise strict monotonicity,

(2) Piecewise smoothness,

(3) $g(\overline{I_k})$ is equal to the union of some $\overline{I_\ell}$'s.

(4) There exists an integer p such that $g^p(\overline{I_k}) = X$ for all k.

The conditions (1)-(3) implies that g is a Markov map. Condition (4) is called aperiodicity or transitivity condition.

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Theorem

Assume that (1)-(4) hold and g is eventually expansive. Then g has an ergodic invariant measure equivalent to Lebesgue measure.

Partition $\mathscr{P}$ for f

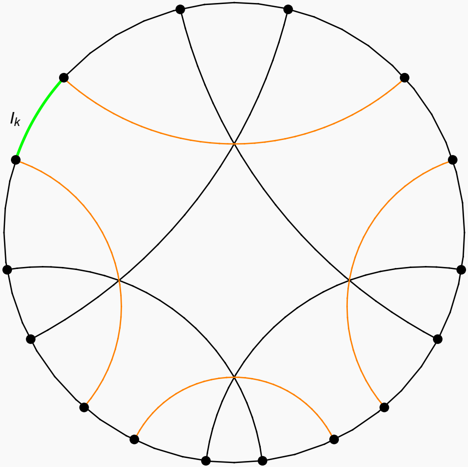

Figure 7: The set E of the end points of the geodesics in N form the partition intervals $\{I_k\}$.

Markov Property for f

Bowen & Series show that f is a Markov function with respect to $\mathscr{P}$. To show (3): $f(\overline{I_k})$ is equal to the union of some $\overline{I_\ell}$'s, it is enough to show that E is invariant under f.

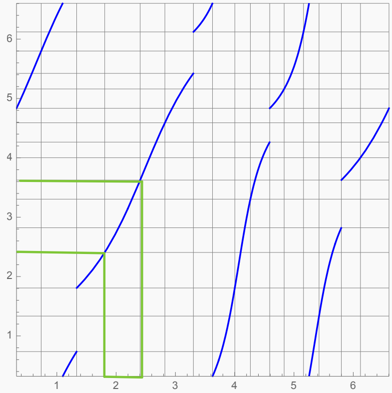

Figure 8: The Bowen-Series function f for the triangle group (6,6,3).

Partition $\mathscr{P}_\alpha$ for f_α

For f_α, we define $\mathscr{P}_\alpha$ using

$$E_\alpha = E \cup \{f_\alpha^k(\alpha)\}_{k\geq 0} \cup \{f_\alpha^k(T_{i-1}(\alpha))\}_{k\geq 0}.$$

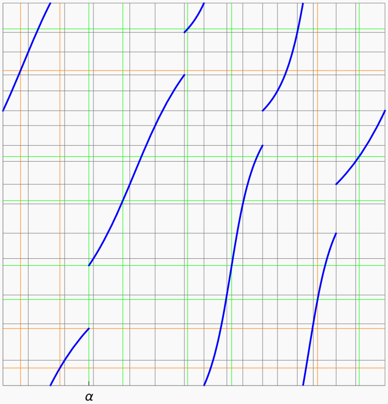

Figure 9: The plot of f_α. Gridlines correspond to the points in E_α.

Markov Property for f_α

E_α is f_α−invariant. Thus, f_α is Markov if E_α is finite.
Since E is finite, we only need that the orbits $\{f_\alpha^k(\alpha)\}_{k\geq 0}$ and $\{f_\alpha^k(T_{i-1}(\alpha))\}_{k\geq 0}$ are finite.

Theorem (Y.K. & Schmidt, 2023)

The function f_α is Markov if and only if α is a hyperbolic fixed point of Γ.

Proof

($\Rightarrow$) If f_α is Markov, then E_α is finite, then f_α-orbit of α is finite, which implies that α is f_α-preperiodic; i.e. there exists $m, n \geq 0$ such that $f_\alpha^m(\alpha) = f_\alpha^n(\alpha)$. Hence, α is a hyperbolic fixed point.

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($\Leftarrow$) Suppose α is a hyperbolic fixed point with infinite f_α-orbit.

Markov Property for f_α

Lemma (Y.K. & Schmidt, 2023)

Fix $\alpha \in \mathscr{O}$. Suppose that $x \in \mathbb{S}^1$ has infinite f_α-orbit. Then there are infinitely many values of j such that the f-orbit of x contains either (1) $f_\alpha^j(x)$, (2) $f \circ f_\alpha^j(x)$, or (3) $f^2 \circ f_\alpha^j(x)$.

Proof (Cont.):

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Proof (Cont.):

- By the Pigeonhole Principle, we can assume one of the three cases occurs for infinitely many values of j.
- One easily shows f-orbit of α is finite.
- Since f is a finite-to-one function and the f-orbit of α is finite, there are only finitely many preimages under f or f^2 of this finite set.
- There are some $m \neq n$ such that $f_\alpha^m(\alpha) = f_\alpha^n(\alpha)$, and so f_α-orbit of α is finite.

Transitivity

Theorem (Y.K. & Schmidt, 2023)

Fix $\alpha \in \mathscr{O}_i$. Let $n_i = |N_i|$. The function f_α is surjective if and only if the following conditions hold:

- $n_i > 2$,
- $n_i = 2$ and $n_{i+2} > 2$,
- α belongs to the closure of the set of points $x \in \mathscr{O}_i$ such that $f^{n_i}(x) = f^{n_i - 1}(T_{i-1}x)$.

Moreover, if f_α is Markov, then f_α has ergodic invariant measure equivalent to Lebesgue measure if and only if f_α is surjective.

Transitivity

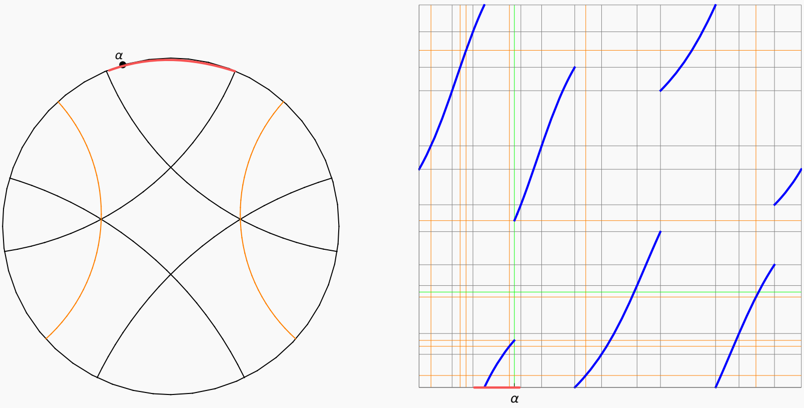

Figure 10: Plot of the function f_α for the signature (4, 4, 3). This function is not transitive; it is not even a surjective function!

Thank you for your attention!

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