Monogenic representation for self-similar random fields and color images

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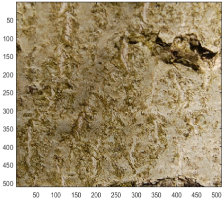
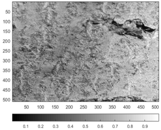
june, 29th 2023, Multifractal analysis and self-similarity, Luminy

joint work with Philippe Carré (XLIM, Poitiers), Céline Lacaux (LMA, Avignon), Claire Launay (IDP, Tours)

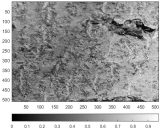
RGB Color textures



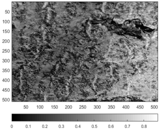
Wood texture from STex databasis



R

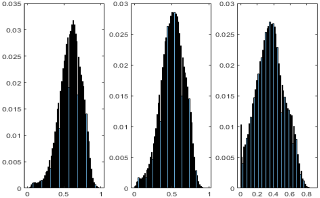


G

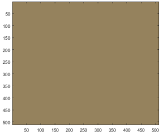


B

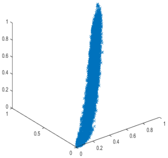
Marginal statistics



RGB histograms



Mean color I_0



RGB values

$$\mathrm{Cor}_{\mathrm{RGB}} = \begin{pmatrix} 1 & 0.9937 & 0.9591 \\ 0.9937 & 1 & 0.9775 \\ 0.9591 & 0.9775 & 1 \end{pmatrix}$$

High correlation between channels

Color wave and random field extension

Following Carré and Soulard [1] the color wave is defined at pixel $x \in \mathbb{R}^2$ as

$$I(x) = I_0 + \begin{bmatrix} A_R \cos(w \cdot x + \varphi_R) \\ A_G \cos(w \cdot x + \varphi_G) \\ A_B \cos(w \cdot x + \varphi_B) \end{bmatrix} = I_0 + \Re\left(\Gamma e^{iw \cdot x}\right),$$

where $\Gamma \in \mathbb{C}^3$ given by

$$\Gamma = \begin{bmatrix} A_R e^{i\varphi_R} \\ A_G e^{i\varphi_G} \\ A_B e^{i\varphi_B} \end{bmatrix},$$

is the **color atom**. We will assume that

$$I(x) = I_0 + \Re(\Gamma Z(x)),$$

for $(Z(x))_{x \in \mathbb{R}^2}$ a complex valued centered Gaussian random field as in Kaseb PhD [2].

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1. *Elliptical monogenic representation of color images and local frequency analysis*, 2015
2. *Parametric and Stochastic Characterization of Color Textures*, 2021

Color atom and fields

Writing $Z(x) = |Z(x)|e^{i\phi_Z(x)} \in \mathbb{C}$ in polar coordinates, for the color atom

$$\Gamma = \begin{bmatrix} A_R e^{i\varphi_R} \\ A_G e^{i\varphi_G} \\ A_B e^{i\varphi_B} \end{bmatrix} \in \mathbb{C}^3,$$

modeling the RGB pixel $x \in \mathbb{R}^2$ by

$$I(x) = I_0 + \Re\left(\Gamma Z(x)\right),$$

implies that for each channel $C \in \{R, G, B\}$

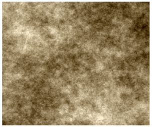
$$\begin{aligned} I_C(x) &= I_{0,C} + \Re\left(A_C|Z(x)|e^{i[\varphi_C + \phi_Z(x)]}\right) \\ &= I_{0,C} + A_C|Z(x)|\cos[\varphi_C + \phi_Z(x)] \end{aligned}$$

- In synthesis we need to define Γ and simulate $(Z(x))_x$;
- In analysis we only have access to $(I_C(x))_x$ for all channels ;

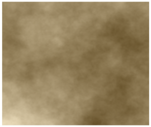
RGB Color textures synthesis

Synthesis from Wood texture RGB 512×512

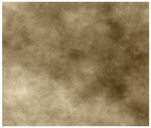
Wood



$H = 0.2$



$H = 0.7$



$H = 0.5$

$H = 0.5$, $\delta = \pi/6$

$H = 0.5$, $\delta = \theta = \pi/6$

Outlines

1 Harmonizable Gaussian random fields
- Gaussian measures and harmonizable representation
- Anisotropic Self-similar fields
- Elementary or Lighthouse fields

2 Riesz transforms and Monogenic signal
- Riesz transforms
- Monogenic random field
- Monogenic parameters and inference

Gaussian random measures

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. Let λ be the Lebesgue measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$

$$\mathcal{E} = \{A \in \mathcal{B}(\mathbb{R}^2) \text{ s.t. } \lambda(A) < +\infty\}.$$

A real Gaussian random measure M is a stochastic process $M = \{M(A); A \in \mathcal{E}\}$ satisfying

- For all $A \in \mathcal{E}$, $M(A)$ is a real Gaussian r.v. $\mathcal{N}(0, \lambda(A))$;
- For $A_1, \ldots, A_n \in \mathcal{E}$ disjoint sets the r.v. $M(A_1), \ldots, M(A_n)$ are $\perp\!\!\!\perp$;
- For $(A_n)_{n\in\mathbb{N}}$ disjoint sets s.t. $\underset{n\in\mathbb{N}}{\cup} A_n \in \mathcal{E}$,

$$M(\underset{n\in\mathbb{N}}{\cup} A_n) = \sum_{n\in\mathbb{N}} M(A_n) \text{ a.s.}$$

Proposition

$f \in L^2_{\mathbb{R}}(\mathbb{R}^2) \mapsto M(f) = \int_{\mathbb{R}^2} f(x) M(dx) \sim \mathcal{N}\left(0, \int_{\mathbb{R}^2} f(x)^2 dx\right)$.

Complex Gaussian measure

Complex Gaussian measure [3] : $W = M_1 + iM_2$ with $M_1 \perp\!\!\!\perp M_2$

Proposition

$g \in L^2(\mathbb{R}^2) \mapsto W(g) = \int_{\mathbb{R}^2} g(\xi)W(d\xi) := \Re(W(g)) + i\Im(W(g))$ *with* $\mathbf{W}(g) = [\Re(W(g))\ \Im(W(g))]^T \sim \mathcal{N}\left(0, \left(\int |g|^2\right) I_2\right)$,

For $g, g' \in L^2(\mathbb{R}^2)$, the covariance is given by

$$\mathbb{E}\left(W(g)\overline{W(g')}\right) = 2\langle g, g'\rangle,$$

and the pseudo-covariance $\mathbb{E}\left(W(g)W(g')\right) = 0$ while

$$\mathbb{E}\left(\mathbf{W}(g)\mathbf{W}(g')^T\right) = \mathbb{E}\begin{pmatrix} \Re(\langle g, g'\rangle) & -\Im(\langle g, g'\rangle) \\ \Im(\langle g, g'\rangle) & \Re(\langle g, g'\rangle)) \end{pmatrix},$$

with

$$\langle g, g'\rangle = \int_{\mathbb{R}^2} g(\xi)\overline{g'(\xi)}d\xi.$$

3. see Samorodnitsky, Taqqu, (1994) for instance

Harmonizable stationary Gaussian random fields

For $g \in L^2(\mathbb{R}^2)$, a stationary harmonizable random field may be defined as

$$Z(x) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi} g(\xi) W(d\xi) = X(x) + iY(x),\ x \in \mathbb{R}^2$$

with for $f = |g|^2 \in L^1_{\mathbb{R}}(\mathbb{R}^2)$ and $\hat{f}(x) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi} f(\xi) d\xi$

$$\mathbb{E}\left(Z(x)\overline{Z(x')}\right) = 2\hat{f}(x - x')$$

while $\mathbb{E}\left(Z(x)Z(x')^T\right) = \mathbb{E}\begin{pmatrix} \Re(\hat{f}(x-x')) & -\Im(\hat{f}(x-x')) \\ \Im(\hat{f}(x-x')) & \Re(\hat{f}(x-x')) \end{pmatrix}$.

Note that for any $\varphi \in \mathbb{R}$

$$e^{i\varphi} Z \overset{fdd}{=} Z$$

and $Z(x) = |Z(x)|e^{i\phi_Z(x)}$ with $\phi_Z(x) \sim \mathcal{U}([-\pi, \pi)) \perp\!\!\!\perp \frac{|Z(x)|^2}{\hat{f}(0)} \sim \chi^2(2)$.

Moreover $X \overset{fdd}{=} Y$ with $\mathrm{Cov}(X(x), X(x')) = \Re(\hat{f}(x - x')) = c_X(x - x')$.

Real Harmonizable stationary random fields

Let $(X(x))_{x\in\mathbb{R}^2}$ be a real stationary centered Gaussian random field with covariance $c_X : \mathbb{R}^2 \to \mathbb{R}$.

Theorem (Bochner 1932)

there exists a symmetric probability measure ν on $\mathbb{R}^2$ such that

$$c_X(x) = c_X(0) \int_{\mathbb{R}^2} e^{-i\xi\cdot x} d\nu(\xi).$$

Assuming that $d\nu(\xi) = f(\xi)d\xi$ one has f even in $L^1_{\mathbb{R}}(\mathbb{R}^2)$ and for g s.t. $|g|^2 = f$, $\Re(Z)$ and $\Im(Z)$ are independent copies of X.

Extension to stationary increments fields (SI)

For $g \in L^2(\mathbb{R}^2, \min(1, |\xi|^2)d\xi)$ we can define

$$Z_0(x) = \int_{\mathbb{R}^2} \left[e^{-ix\cdot\xi} - 1\right] g(\xi)W(d\xi),\ x \in \mathbb{R}^2.$$

Then $X = \Re(Z_0)$ has now stationary increments (SI)

$$\forall x_0 \in \mathbb{R}^2,\ \{X(x + x_0) - X(x_0)\,;\, x \in \mathbb{R}^2\} \overset{fdd}{=} \{X(x)\,;\, x \in \mathbb{R}^2\}.$$

Its **spectral density** is given by $f(\xi) = |g(\xi)|^2 \in L^1_{\mathbb{R}}(\mathbb{R}^2, \min(1, |\xi|^2)d\xi)$.

Choosing f even $\Im(Z_0)$ is also an independent copy of X but now

$$Z_0(x) = |Z_0(x)|e^{i\phi_{Z_0}(x)} \text{ with } \phi_{Z_0}(x) \sim \mathcal{U}([-\pi, \pi)) \perp\!\!\!\perp \frac{|Z(x)|^2}{v_X(x)} \sim \chi^2(2)$$

$$\text{for } v_X(x) = \mathrm{Var}(X(x)) = \int_{\mathbb{R}^2} \left|e^{-ix\cdot\xi} - 1\right|^2 f(\xi)d\xi = \frac{1}{2}\mathbb{E}(|Z_0(x)|^2).$$

Rk : when $f \in L^1_{\mathbb{R}}(\mathbb{R}^2)$ one has $Z_0 = Z - Z(0)$.

Anisotropic self-similar fields

For $H \in (0,1)$ take $f(\xi) = t(\xi/|\xi|)|\xi|^{-2H-2}$, for $t \in L^1_{\mathbb{R}}(S^1)$ even non-negative. Then X is SSI of order H

$$\{X(\lambda x); x \in \mathbb{R}^2\} \overset{fdd}{=} \lambda^H \{X(x); x \in \mathbb{R}^2\}.$$

Main Properties :

- H a.s. critical Hölder exponent [4]
- H a.s. fractal dimension [5] :

$$\dim_{\mathcal{H}} \left(\{(x, X(x)), x \in [0,1]^2\}\right) = 3 - H \text{ a.s.}$$

- For any $x \in \mathbb{R}^2$ and $\Theta \in S^1$, $\{X(x + s\Theta); s \in \mathbb{R}\}$ is a fBm of order H
- when t is constant, $X \circ R \overset{fdd}{=} X$ for all rotation R and X is isotropic called (Lévy) fractional Brownian field

4. Bonami, Estrade, *Anisotropic analysis of some Gaussian models*, 2003
5. Xiao, *Sample path properties of anisotropic Gaussian random fields*, 2009

Elementary or lighthouse fields

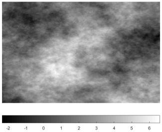
Elementary fields[6] For $H \in (0,1)$ and $\delta \in (0, \pi/2]$, define

$$X(x) = \Re\left(\int_{\mathbb{R}^2} [e^{-ix\cdot\xi} - 1]\sqrt{f(\xi)}W(d\xi)\right),$$

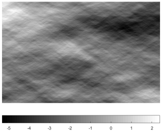
for $f(\xi) = t_\delta(\xi/|\xi|)|\xi|^{-2H-2}$, with t_δ even given for $\alpha \in (-\pi/2, \pi/2]$ and $\Theta(\alpha) = (\cos(\alpha), \sin(\alpha)) \in S^1$ by

$$t_\delta(\Theta(\alpha)) = 1_{|\alpha|\le\delta}.$$

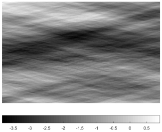
Then $\mathrm{Cov}(X(x), X(y)) = \frac{1}{2}(v_X(x) + v_X(y) - v_X(x-y))$ with $v_X(x) = c_\delta(x/|x|)|x|^{2H}$.



$\delta = \pi/2$ isotropic



$\delta = \pi/3$



$\delta = \pi/6$

6. Biermé, Moisan, Richard, *A Turning-Band Method for the Simulation of Anisotropic Fractional Brownian Fields*, 2015

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Riesz transforms

For a real function $u \in L^2_{\mathbb{R}}(\mathbb{R}^2)$ the Riesz transforms are defined as singular integrals for $k = 1, 2$

$$\mathcal{R}_k(u)(x) = \frac{1}{2\pi} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^2 \setminus B_\varepsilon(x)} \frac{(x_k - t_k)}{|x - t|^3} u(t) dt.$$

Proposition (see Stein (1993) for instance)

- $\mathcal{R}_k : L^2_{\mathbb{R}}(\mathbb{R}^2) \to L^2_{\mathbb{R}}(\mathbb{R}^2)$ *continuous ;*
- *Fourier multiplier* $\widehat{\mathcal{R}_k(u)}(\xi) = -i \frac{\xi_k}{|\xi|} \hat{u}(\xi)$ *;*
- $\langle \mathcal{R}_k u, v \rangle = -\langle u, \mathcal{R}_k v \rangle$ *;*
- $\mathcal{R}_1^2 + \mathcal{R}_2^2 = -I$.

It follows

- **Translation invariance** : $\mathcal{R}_k \tau_x u = \tau_x \mathcal{R}_k u$;
- **Scale invariance** : $\mathcal{R}_k \delta_\lambda u = \delta_\lambda \mathcal{R}_k u$;

with $\tau_x u(y) = u(y - x)$ and $\delta_\lambda u(y) = u(\lambda y)$.

Generalized field point of view

Recall that the *Schwartz space* $\mathcal{S}_0 = \mathcal{S}_0(\mathbb{R}^2)$ consists of functions $u : \mathbb{R}^2 \to \mathbb{R}$ in $\mathcal{C}^\infty(\mathbb{R}^2)$ s.t. $\int_{\mathbb{R}^2} u(x)dx = 0$ and

$$\forall m \in \mathbb{N} \text{ and } j = (j_1, j_2) \in \mathbb{N}^2,\ \|u\|_{m,j} = \sup_{x\in\mathbb{R}^2} (1+|x|)^m \left|D^j u(x)\right| < \infty.$$

Let $X = \Re(Z)$ with Z harmonizable for $f \in L^1(\mathbb{R}^2, \min(1,|\xi|^2)d\xi)$ and define the generalized fields (random elements of $(\mathcal{S}_0', \mathcal{B}(\mathcal{S}_0'))$ [7] as

- $\langle X, u\rangle = \Re\left(\langle Z, u\rangle\right) = \Re\left(\int_{\mathbb{R}^2} \hat{u}(\xi)\sqrt{f(\xi)}W(d\xi)\right)$ $\sim \mathcal{N}\left(0, \int_{\mathbb{R}^2} |\hat{u}(\xi)|^2 f(\xi)d\xi\right)$;
- $\langle \mathcal{R}_k X, u\rangle = -\langle X, \mathcal{R}_k u\rangle = \Re\left(\int_{\mathbb{R}^2} \frac{i\xi_k}{|\xi|}\hat{u}(\xi)\sqrt{f(\xi)}W(d\xi)\right)$ $\sim \mathcal{N}\left(0, \int_{\mathbb{R}^2} |\hat{u}(\xi)|^2 \frac{\xi_k^2}{|\xi|^2} f(\xi)d\xi\right)$;

7. Biermé, Durieu, Wang, *Generalized random fields and Lévy's continuity theorem*, 2018

Monogenic random field

The Monogenic[8] random field is given by the $\mathbb{R}^3$-valued generalized random field with for $u \in \mathcal{S}_0(\mathbb{R}^2)$

$$MX(u) = (\langle X, u\rangle, \mathcal{R}_X(u)) \text{ where } \mathcal{R}_X(u) = (\langle \mathcal{R}_1 X, u\rangle, \langle \mathcal{R}_2 X, u\rangle).$$

It follows that for u even, one has $\langle X, \tau_x u\rangle = X * u(x)$, the pointwise random field $(MX(\tau_x u))_{x\in\mathbb{R}^2}$ is Gaussian centered stationary with covariance given by

$$\mathbb{E}\left(MX(\tau_x u)MX(u)^T\right) = \Re \int \begin{pmatrix} 1 & -i\frac{\xi_1}{|\xi|} & -i\frac{\xi_2}{|\xi|} \\ i\frac{\xi_1}{|\xi|} & \frac{\xi_1^2}{|\xi|^2} & \frac{\xi_1\xi_2}{|\xi|^2} \\ i\frac{\xi_2}{|\xi|} & \frac{\xi_1\xi_2}{|\xi|^2} & \frac{\xi_2^2}{|\xi|^2} \end{pmatrix} e^{-ix\cdot\xi}|\hat{u}(\xi)|^2 f(\xi)d\xi.$$

Easily obtain a multiscale representation by choosing $u \in \mathcal{S}_0(\mathbb{R}^2)$ with $\int u^2 = 1$ and defining

$$u_j(y) = 2^{-j}u(2^{-j}y), y \in \mathbb{R}^2, j \in \mathbb{Z}.$$

Rk : due to translation invariance of Riesz transform we also have

$$\text{Cov}\left(\mathcal{R}_1 X(\tau_x u), \mathcal{R}_2 X(u)\right) = \text{Cov}\left(\mathcal{R}_1 X(u), \mathcal{R}_2 X(\tau_x u)\right).$$

8. M. Felsberg and G. Sommer, *The monogenic signal*, 2001

Monogenic representation for lighthouse fields

Consider $(\delta, H) \in (0, \pi/2] \times (0,1)$ and $u \in \mathcal{S}_0(\mathbb{R}^2)$ a radial function. If X is an elementary field with spectral density $f_X(\xi) = t_\delta(\xi/|\xi|)|\xi|^{-2H-2}$ then $(MX(\tau_x u))_{x\in\mathbb{R}^2}$ is a centered stationary Gaussian field such that for all $x \in \mathbb{R}^2$,

$$MX(\tau_x u) \overset{d}{=} \sqrt{c_X(u)} D_\delta Z,$$

where

- $c_X(u) = \mathrm{Var}(\langle X, u\rangle) = \int_{\mathbb{R}^2} |\widehat{u}(\xi)|^2 f(\xi) d\xi$,
- $D_\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{1}{2}(1+\chi_X)} & 0 \\ 0 & 0 & \sqrt{\frac{1}{2}(1-\chi_X)} \end{pmatrix}$ where $\chi_X = \frac{\sin(2\delta)}{2\delta}$;
- $Z \sim \mathcal{N}(0, I_3)$.

Moreover, for a given scale j,

$$(MX(\tau_x u_j))_{x\in\mathbb{R}^2} \overset{fdd}{=} 2^{j(H+1)} (MX(\tau_{2^{-j}x} u))_{x\in\mathbb{R}^2}.$$

Monogenic parameters

Using spherical coordinates[9] one has

$$MX(\tau_x u) = A(\tau_x u)\left(\cos(\varphi(\tau_x u)), \sin(\varphi(\tau_x u))\cos(\theta(\tau_x u)), \sin(\varphi(\tau_x u))\sin(\theta(\tau_x u))\right),$$

such that

$$\langle X, \tau_x u\rangle = \Re\left(A(\tau_x u)e^{i\varphi(\tau_x u)}\right).$$

Here we have,

- $A(\tau_x u) = |MX(\tau_x u)|$ amplitude ;
- $\varphi(\tau_x u) \in [0, \pi]$ phase ;
- $\theta(\tau_x u) \in (-\pi, \pi]$ orientation.

9. R. Soulard et P. Carré, *Characterization of color images with multiscale monogenic maxima*, 2017.

Distribution of the spherical coordinates

- For all $x \in \mathbb{R}^2$, $(A(\tau_x u), \theta(\tau_x u), \varphi(\tau_x u)) \stackrel{d}{=} (A(u), \theta(u), \varphi(u))$.

- For all $j \in \mathbb{Z}$, $(A(u_j), \theta(u_j), \varphi(u_j)) \stackrel{d}{=} \left(2^{j(H+1)} A(u), \theta(u), \varphi(u)\right)$

- **Orientation distribution :** $\theta(u)$ follows an offset normal distribution whose probability density function is π-periodic and given by

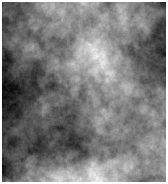
$$\alpha \mapsto \frac{\sqrt{1-\chi_X^2}}{2\pi(1-\chi_X \cos(2\alpha))},$$

where $\chi_X = \frac{\sin(2\delta)}{2\delta} \in [0,1)$ is the coherence index.

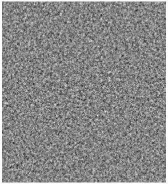
- **Isotropic case :** if $\delta = \pi/2$, $\chi_X = 0$, $\theta(u)$ follows a uniform distribution on $(-\pi, \pi)$ and is independent of $(A(u), \varphi(u))$. Moreover, the density function of the phase $\varphi(u)$ is given by

$$\phi \mapsto \frac{|\sin(\phi)|}{(1+\sin(\phi)^2)^{3/2}} 1_{[0,\pi]}(\phi).$$

Monogenic representation of an elementary field



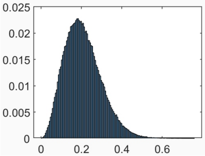
Random field



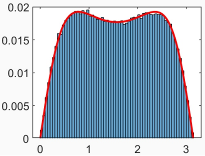
Filtered field

Riesz 1

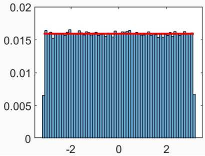
Riesz 2



Amplitude



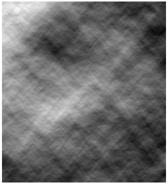
Phase



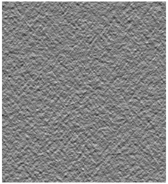
Orientation

Elementary field with $H = 0.5, \delta = \pi/2$.

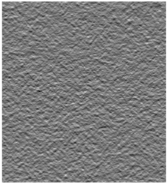
Monogenic representation of an elementary field



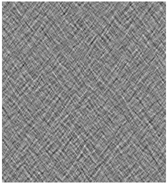
Random field



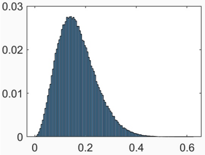
Filtered field



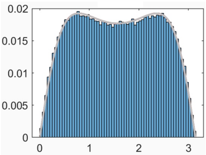
Riesz 1



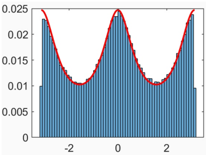
Riesz 2



Amplitude



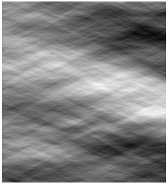
Phase



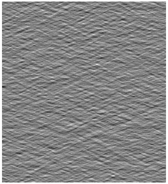
Orientation

Elementary field with $H = 0.5, \delta = \pi/3$.

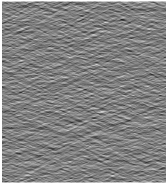
Monogenic representation of an elementary field



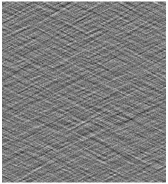
Random field



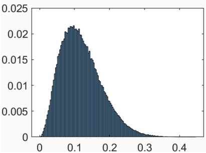
Filtered field



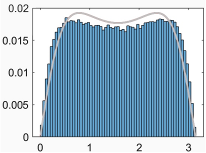
Riesz 1



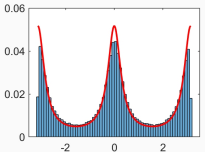
Riesz 2



Amplitude



Phase



Orientation

Elementary field with $H = 0.5, \delta = \pi/6$.

Riesz structure tensor and coherence index

Following Ohlede et al [10], Polisano [11] proposes to use **Riesz structure tensor**

$$J_X(u) := \mathbb{E}\left(\mathcal{R}X(u)\mathcal{R}X(u)^T\right).$$

Let $\lambda^+(u)$ the largest eigenvalue of $J_X(u)$ and $\lambda^-(u)$ the smallest one Then the **coherence index**

$$\chi_X(u) = \frac{\lambda^+(u) - \lambda^-(u)}{\lambda^+(u) + \lambda^-(u)} \in [0,1).$$

allows to measure directional anisotropy.
Note that, for lighthouse field and radial $u \in \mathcal{S}_0(\mathbb{R}^2)$ one has for $f(\xi) = t_\delta(\xi/|\xi|)|\xi|^{-2H-2}$

$$J_X(u) = c_X(u) \times \operatorname{diag}\left(\frac{1}{2}(1+\chi_X), \frac{1}{2}(1-\chi_X)\right)$$

Hence

$$\lambda^\pm(u) = c_X(u) \times \left[\frac{1}{2} \pm \frac{\sin(2\delta)}{4\delta}\right] \text{ and } \chi_X(u) = \frac{\sin(2\delta)}{2\delta} = \chi_X.$$

10. *Detecting directionality in random fields using the monogenic signal*, 2014
11. *Modélisation de textures anisotropes par la transformée en ondelettes monogéniques*, PhD Thesis, 2017

Inference using Riesz structure tensor

We observe the values on a pixel grid $G_N \subset \mathbb{Z}^2$ of size $N \times N$ for some scales j.
Empirical estimator of the structure tensor :

$$J_{\text{emp}}(u_j) = \frac{1}{N^2} \sum_{x \in G_N} \mathcal{R}X(\tau_x u_j)\mathcal{R}X(\tau_x u_j)^*,$$

with $(\lambda^+_{\text{emp}}(u_j), \lambda^-_{\text{emp}}(u_j))$ its largest and smallest eigenvalues.

Proposition $J_{\text{emp}}(u_j)$ is an unbiased estimator and

$$J_{\text{emp}}(u_j) \xrightarrow[N\to\infty]{a.s.} J_X(u_j)$$

with asymptotic normality [12] for $H < 1/2$ or choosing $u \in \mathcal{S}_0(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} x_k u(x)dx = 0$ for $k = 1, 2$ when $H \in [1/2, 1)$.

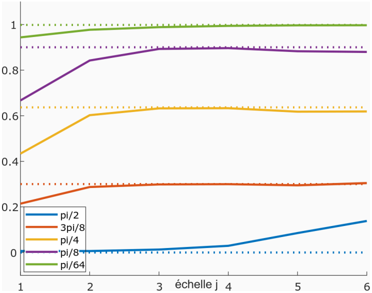
Besides, $\lambda^{\pm}_{\text{emp}}(u_j) \xrightarrow[N\to\infty]{a.s.} \lambda^{\pm}(u_j)$ with asymptotic normality and similarly

$$\chi_{\text{emp}}(u_j) := \frac{\lambda^+_{\text{emp}}(u_j) - \lambda^-_{\text{emp}}(u_j)}{\lambda^+_{\text{emp}}(u_j) + \lambda^-_{\text{emp}}(u_j)} \xrightarrow[N\to\infty]{a.s.} \chi_X.$$

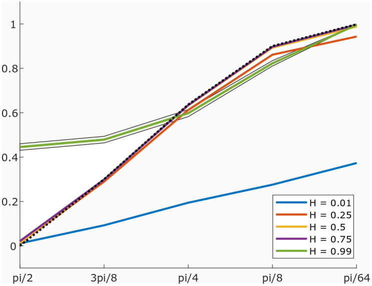
12. Arcones *Limit theorems for nonlinear functionals of a stationary Gaussian sequence of vectors, 1994*

Inference using the structure tensor - Coherence index

$$\chi x(u_j) = \frac{\sin(2\delta)}{2\delta}$$



a) $H = 0.5$



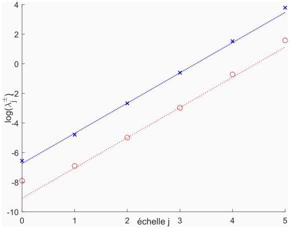
b) $j = 3$

Coherence index estimation, for 1000 realizations, depending on

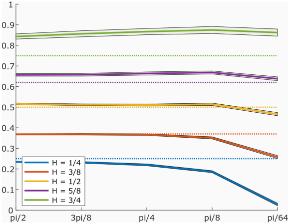
a) the scale of the monogenic representation j and the degree of anisotropy δ, with $H = 0.5$ and
b) the degree of anisotropy and H, with $j = 3$.

Inference using the structure tensor - Hurst index

$$\lambda^{\pm}(u_j) = 2^{j(2H+2)}\lambda^{\pm}(u)$$



a) $\lambda^{\pm}_{\mathrm{emp}}(u_j)$



b) Estimation of H

Estimation of H, obtained by the eigenvalues $\lambda_j^{\pm\mathrm{emp}}$ from $J_X(u_j)$

a) Dotted lines are computed for scale $j = 2$ and $j + 1 = 3$ and provide estimators of the Hurst parameter H

$$H^{\pm}_{\mathrm{emp}}(u_j) = \frac{1}{2}\log_2(\lambda^{\pm}_{\mathrm{emp}}(u_{j+1})/\lambda^{\pm}_{\mathrm{emp}}(u_j)) - 1.$$

b) Depending on H and δ.

Inference using the monogenic signal

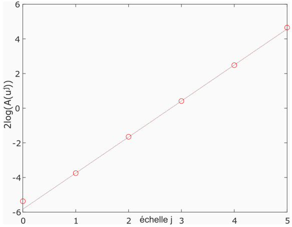
Squared amplitude : $A(u_j)^2 = \langle X, u_j \rangle^2 + |\mathcal{R}X(u_j)|^2$ and $A(u_j)^2 \overset{d}{=} c_X(u_j) A^2$, with $A^2 = |D_\delta Z|^2$ for $Z \sim \mathcal{N}(0, I_3)$, D_δ and

$$c_X(u_j) = \mathrm{Var}(\langle X, u_j \rangle) = 2^{j(2H+2)} \mathrm{Var}(\langle X, u \rangle) = 2^{j(2H+2)} c_X(u).$$

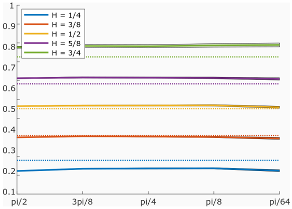
Proposition The estimator $V_j^{emp} = \frac{1}{N^2} \sum_{x \in G_N} A(\tau_x u_j)^2$ is unbiased and

$$V_j^{emp} \xrightarrow[N \to \infty]{a.s.} \mathbb{E}\left(A(u_j)^2\right)$$

with asymptotic normality for $H < 1/2$ or choosing $u \in \mathcal{S}_0(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} x_k u(x) dx = 0$ for $H \in [1/2, 1)$.



c) V_j^{emp}



d) Estimation of H

Conclusion

- New estimators based on Riesz transform and Monogenic signal for SSI anisotropic fields
- Monogenic signal seems to produce more stable estimators
- Interesting to statistically evaluate multiscale monogenic algorithm (used of undecimated filtered bank)

Perspectives :

- Study further the statistical properties of these estimators and choice of filters
- Define "mean" features for angles
- Extend to color textures ie multivariate random fields : use elliptical model to estimate color atom...

☛ **Postdoc position available**

Many thanks for your attention !

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