Fractional Gaussian and Stable fields on fractals

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Outline

1. Fractional Brownian Motion on $\mathbb{R}^n$
   - Definition and Smoothness
   - Link with fractional Laplacian
2. Sierpiński gasket and fractional Laplace operator
3. Fractional Gaussian fields on Sierpiński Gasket
4. Stable fractional random fields

Gaussian random fields

Let us consider a probablity space $(\Omega, \mathcal{F}, \mathbb{P})$, **a non-empty set U** and a collection $\boldsymbol{X = (X(x))_{x\in U}}$ of **real-valued random variables** indexed by U, that is s.t. for all $x \in U$,

$$X(x): \begin{array}{ccc} \Omega & \longrightarrow & \mathbb{R} \\ \omega & \longmapsto & X(x,\omega) \end{array}$$

is a measurable function.

Definition

The collection $\boldsymbol{X}$ is a **Gaussian random field on U** iff for any $n \in \mathbb{N}^*$, for any $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and for any $x_1, \ldots, x_n \in U$, the linear combination

$$\sum_{i=1}^{n} \lambda_i X(x_i)$$

is a **Gaussian random variable.**

Gaussian random fields

- **The distribution of a Gaussian random field X** is **characterized by its mean function**

$$m_X : x \mapsto \mathbb{E}(X(x))$$

and its covariance function

$$\Gamma_X : (x, y) \longmapsto \mathbb{E}(X(x)X(y)) - m_X(x)m_X(y).$$

- **The Gaussian field X is centered** when $m_X \equiv 0$.

- **The sample paths regularity of a Gaussian random field X**, that is of

$$x \longmapsto X(x, \omega),$$

is **linked with the behaviour of**

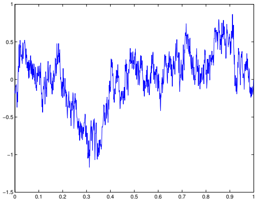
$$\mathbb{E}\Big([X(x) - X(y)]^2\Big) = \Gamma_X(x, x) + \Gamma_X(y, y) - 2\Gamma_X(x, y).$$

Fractional Brownian Motion on $\mathbb{R}^n$

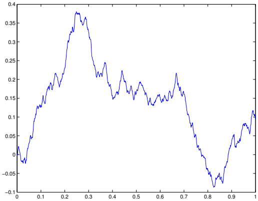
Kolmogorov 1940 / Mandelbrot, Van Ness 1968

Let $H \in (0,1)$. **The FBM $(B_H(x))_{x\in\mathbb{R}^n}$** is the real **centered Gaussian random field** with covariance function

$$\boldsymbol{\Gamma_{B_H}(x,y) = \frac{1}{2}\left(\|x\|^{2H} + \|y\|^{2H} - \|x-y\|^{2H}\right).}$$



$H = 0.3$



$H = 0.8$

Sample paths regularity: modulus of continuity

- For the FBM B_H,

$$\mathbb{E}\Big((B_H(x) - B_H(y))^2\Big) = \|x - y\|^{2H}.$$

- Then, there exists a **modification of B_H**, that is **a random field B_H^*** such that

$$\textbf{for all } x \in \mathbb{R}^n,\ B_H^*(x) = B_H(x) \quad \textbf{almost surely,}$$

and for any compact $K \subset \mathbb{R}^n$,

$$\lim_{\delta \to 0} \sup_{\substack{\|x-y\| \leqslant \delta \\ x,y \in K}} \frac{\left|B_H^*(x) - B_H^*(y)\right|}{\|x - y\|^H \sqrt{|\ln \|x - y\||}} < \infty.$$

- **Sample paths**
  - **Locally hölderian of order $\gamma < H$.**
  - **Optimal result**

Outline

Gaussian random measure and stochastic integrals

Gaussian random measure with intensity μ

Let μ a σ-finite measure on $(E, \mathcal{A})$. **The random measure W** is a **Gaussian measure with intensity μ** when

- **for any measurable pairwise disjoint sets $A_n, n \in \mathbb{N}$,**

$$W\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} W(A_n) \text{ almost surely ;}$$

- **for any measurable pairwise disjoint sets $A_1, \ldots, A_n$**, the Gaussian random variables $W(A_1), \ldots, W(A_n)$ are **independent** ;
- for any measurable set $A \in \mathcal{A}$ s.t. $\mu(A) < +\infty$, $W(A)$ is a **centered Gaussian random variable with variance $\mu(A)$**.

Gaussian random measure and stochastic integrals

Let W **be a Gaussian measure on** E **with intensity** μ.

- Then, for any real-valued function $f \in L^2(E, \mathcal{A}, \mu)$, the stochastic integral

$$W(f) := \int_E f(x)\, W(\mathrm{d}x)$$

is **well-defined** and is a **centered Gaussian random variable** with variance

$$\mathrm{Var}\, W(f) = \mathbb{E}\Big(W(f)^2\Big) = \int_E |f(x)|^2\, \mu(\mathrm{d}x).$$

Fractional Gaussian fields on $\mathbb{R}^n$ using fractional Laplacian

Definition (A. Lodhia, S. Sheffield, X. Sun, and S. S. Watson)

Let us consider

- W a **Gaussian random measure on** $\mathbb{R}^n$ with $\mu = \lambda_n$ **the Lebesgue measure as intensity**
- and $\boldsymbol{\Delta}$ **the Laplace operator on** $\mathbb{R}^n$.

For $s \geqslant 0$, a **fractional Gaussian field** X_s is then defined by

$$\boldsymbol{X_s = (-\Delta)^{-s} W}$$

which means that for any f is the Schwartz space $\mathcal{S}(\mathbb{R}^n)$,

$$\langle X_s, f \rangle = \int_{\mathbb{R}^n} (-\Delta)^{-s} f(x)\, W(\mathrm{d}x)$$

Fractional Gaussian fields on $\mathbb{R}^n$ using fractional Laplacian

- If $s \in \left(\frac{n}{4}, \frac{n}{4} + \frac{1}{2}\right)$, then

$$\langle X_s, f \rangle = \int_{\mathbb{R}^n} f(x) B_H(x) \mathrm{d}x$$

where B_H is the **fractional Brownian motion of order $H = 2s - \frac{n}{2} \in (0, 1)$.**

  - B_H is **the density of X_s** with respect to the Lebesgue measure.

Fractional Gaussian fields on fractals

- We adopt this point of view: **for K a fractal set equipped with a measure μ,** we define **the Fractional Gaussian field on K** as **the Gaussian random measure**

$$X_s = (-\Delta)^{-s} W$$

where Δ is the **Laplace operator on K** and W a **Gaussian random measure with intensity μ.**

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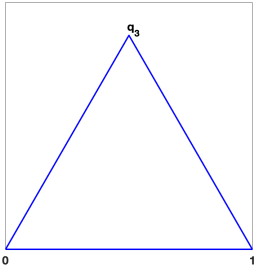
- We focus on
  - K = **Sierpiński gasket,** equipped with the Euclidean metric d and **its Hausdorff measure μ** ;

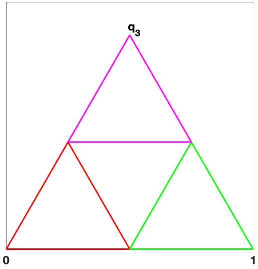
Outline

Sierpiński Gasket

In $\mathbb{R}^2 \sim \mathbb{C}$, consider the triangle with vertices $q_1 = 0, q_2 = 1$ and $q_3 = e^{i\frac{\pi}{3}}$ and the maps

$$F_i : z \longmapsto \frac{1}{2}(z - q_i) + q_i, \quad i \in \{1, 2, 3\}.$$

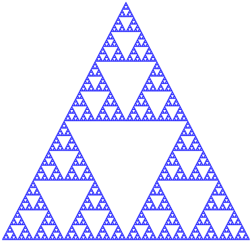




Definition

The Sierpiński gasket is the **unique non-empty compact** set $K \subset \mathbb{C}$ such that

$$\boldsymbol{K = \bigcup_{i=1}^{3} F_i(K).}$$

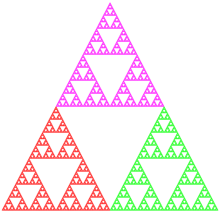


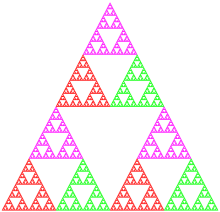
Hausdorff measure on Sierpiński Gasket

Hausdorff measure

The Hausdorff measure μ is the **unique Borel probability on K** such that for each positive integer n and each $(i_1, \ldots, i_n) \in \{1, 2, 3\}^n$,

$$\boldsymbol{\mu(F_{i_1} \circ \cdots \circ F_{i_n}(K)) = 3^{-n}.}$$





Hausdorff measure on Sierpiński Gasket

Key property: Alhfors regularity

The measure μ is **d_h-Alfors regular,** that is there exist $c, C > 0$ such that for every $x \in K$ and every $r \in [0,1]$,

$$cr^{d_h} \leqslant \mu(B(x,r)) \leqslant Cr^{d_h}$$

where $B(x,r)$ the Euclidean ball of center x and radius r.

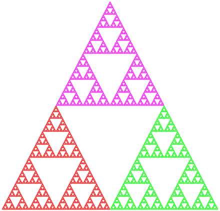
- $d_h = \frac{\ln 3}{\ln 2}$ is **the Hausdorff dimension** of K.

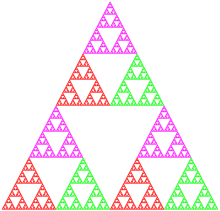
Laplace operator on Sierpiński Gasket

- Let $V_0 = \left\{1, 0, e^{i\frac{\pi}{3}}\right\}$ and for $f \in C(K)$

$$\mathcal{E}_n(f,f) = \frac{1}{2}\left(\frac{5}{3}\right)^n \sum_{i_1,\ldots,i_n} \sum_{x,y\in V_i} (f(x)-f(y))^2$$

where $V_i = F_{i_1} \circ \cdots \circ F_{i_n}(V_0)$





Neumann Laplace operator on Sierpiński Gasket

- Then let

$$\mathcal{F} = \left\{ f \in C(K) \,/\, \lim_{n \to +\infty} \mathcal{E}_n(f,f) < +\infty \right\}$$

and for $f \in \mathcal{F}$, consider

$$\mathcal{E}(f,f) = \lim_{n \to +\infty} \mathcal{E}_n(f,f).$$

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Kigami

- $(\mathcal{E}, \mathcal{F})$ is a **local regular Dirichlet form** on $L^2(K, \mu)$
- Its **generator** $\boldsymbol{\Delta}$ is called **Neumann Laplacian** on K:

$$\mathcal{E}(f,f) = -\langle \Delta f, f \rangle.$$

Fractional Laplace operator on Sierpiński Gasket

- Let $L^2_0(K,\mu) = \left\{ f \in L^2(K,\mu) \, / \, \int_K f \mathrm{d}\mu = 0 \right\}$.
- Let $0 < \lambda_1 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$ the **eigenvalues of** $-\Delta$ and $(\Phi_j)_j$ **an orthonormal basis** of $L^2_0(K,\mu)$ such that

$$\Delta \Phi_j = -\lambda_j \Phi_j.$$

- Then for $f \in L^2_0(K,\mu)$ and for $s > 0$,

$$(-\Delta)^{-s} f = \sum_{j=1}^{+\infty} \frac{1}{\lambda_j^s} \left(\int_K \Phi_j(y) f(y) \, \mu(\mathrm{d}y) \right) \Phi_j$$

is well-defined and $(-\Delta)^{-s} f \in L^2_0(K,\mu)$.

Dirichlet forms & Semi-groups

- For $\mathcal{E}$ a Dirichlet form with generator Δ,

$$\mathcal{E}(f,f) = \lim_{t\to 0_+} \frac{1}{t}\langle (I - P_t)f, f\rangle$$

with $(P_t)_t$ a semi-group.

- There exists a $(p_t)_t$, called **heat kernel** such that for almost all $x \in K$,

$$P_t f(x) = \int_K f(y)\, p_t(x,y)\, \mu(\mathrm{d}y),\ t > 0.$$

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- The heat kernel $p_t(x,y)$ has sub-Gaussian estimates (Barlow-Perkins, 1988): for $t \in (0,1)$,

$$p_t(x,y) \asymp t^{-\frac{d_h}{d_w}} \exp\left(-c_4 \left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w - 1}}\right)$$

where $d_w = \frac{\ln 5}{\ln 2} > 2$.

Outline

Existence of a density and sample paths regularity

Baudoin, L., 2022

- If $s \in \left(\frac{d_h}{2d_w}, 1 - \frac{d_h}{2d_w}\right)$, then the Gaussian random measure $X_s = (-\Delta)^{-s} W$ admits a density $\widetilde{X}_s$ w.r.t μ, that is

$$\langle X_s, f\rangle = \int_K (-\Delta)^{-s} f(x) W(\mathrm{d}x) = \int_K f(x) \widetilde{X}_s(x) \mu(\mathrm{d}x).$$

  Moreover, $\widetilde{X}_s$ **has a continuous Hölder modification of any order** $\gamma < s d_w - \frac{d_h}{2} := H$.

- $H \in (0, d_w - d_h)$

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- $H \in (0, d_w - d_h)$
- Same results for the Dirichlet Laplacian, see Baudoin & Chen, 2023
- Existence of a density for larger s, see Baudoin & Li, 2023

Existence of a density

- We first consider G_s the fractional Riesz kernel:

$$\boldsymbol{G_s(x,y) = \frac{1}{\Gamma(s)} \int_0^{+\infty} t^{s-1} (p_t(x,y) - 1) \, \mathrm{d}y}$$

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- For $\boldsymbol{s > \frac{d_h}{2d_w}}$, $\boldsymbol{G_s(x,\cdot) \in L^2_0(K,\mu)}$ and if $f \in L^2_0(K,\mu)$,

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- For $s > \frac{d_h}{2d_w}$, $G_s(x,\cdot) \in L_0^2(K,\mu)$ and if $f \in L_0^2(K,\mu)$, then for μ almost x,

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.

- The **density of the fractional Gaussian field** is then

$$\widetilde{X}_s(x) = \int_K G_s(x,y) W(\mathrm{d}y)$$

Sample paths regularity

- When $s < 1 - \frac{d_h}{2d_w}$,

$$|\mathcal{G}_s f(x) - \mathcal{G}_s f(y)| \leqslant C d(x,y)^{s d_w - \frac{d_h}{2}} \|f\|_{L^2_0(K,\mu)}$$

which implies that

$$\begin{aligned} \mathbb{E}\left(\left(\widetilde{X}_s(x) - \widetilde{X}_s(y)\right)^2\right) &= \int_K \left(G_s(x,z) - G_s(y,z)\right)^2 \mu(\mathrm{d}z) \\ &\leqslant C d(x,y)^{2 s d_w - d_h} \end{aligned}$$

Sample path regularity

As a consequence, using classical entropy methods for Gaussian fields, an upper bound of the modulus of continuity is derived, from which Hölder continuity follows.

Baudoin, L.

- Let $s \in \left(\frac{d_h}{2d_w}, 1 - \frac{d_h}{2d_w}\right)$. Then, there exists a modification $\widetilde{X}_s^*$ of $\widetilde{X}_s$

$$\lim_{\delta \to 0} \sup_{\substack{d(x,y) \leqslant \delta \\ x,y \in K}} \frac{\left|\widetilde{X}_s^*(x) - \widetilde{X}_s^*(y)\right|}{d(x,y)^H \sqrt{|\ln d(x,y)|}} < \infty.$$

where $sd_w - \frac{d_h}{2} := H$.

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where $sd_w - \frac{d_h}{2} := H$.

An open question: is-it optimal?

Generalization

- K a fractional metric space (Barlow, 1998):
  - (K, d, μ) compact metric space isometrically embedded in some Euclidean space
  - μ Hausdorff measure on K
  - $(\mathcal{E}, \mathcal{F})$ local regular form on $L^2(K, \mu)$
  - sub-Gaussian estimates for the heat kernel associated with $(\mathcal{E}, \mathcal{F})$
  - midpoint property:

$$\forall x, y \in K, \exists z \in K,\ d(x, z) = d(z, y) = \frac{d(x, y)}{2}.$$

Outline

Stable random measure

- Replace now $\boldsymbol{W}$ by **a symmetric α-stable random measure W_α with control measure μ**:

$$\langle \boldsymbol{X_s}, \boldsymbol{f} \rangle = \int_{\boldsymbol{K}} (-\boldsymbol{\Delta})^{-s} \boldsymbol{f}(\boldsymbol{x}) \boldsymbol{W_\alpha}(\boldsymbol{dx})$$

  with Δ the Dirichlet or Neumann Laplacian on K.

- Here $\langle X_s, f \rangle$ is a symmetric stable random variable:

$$\mathbb{E}\left(\mathrm{e}^{iu\langle X_s, f\rangle}\right) = \exp\left(-|u|^\alpha \int_K |f(x)|^\alpha \mu(\mathrm{d}x)\right), \; u \in \mathbb{R}.$$

Existence of a density

Baudoin, L. In progress, 2023

Let $\alpha \in (0,2)$. Then for $s > \max\left(\frac{(\alpha-1)d_h}{\alpha}, 0\right)$, the random field

$$\widetilde{X}_s(x) = \int_K G_s(x,y) W_\alpha(\mathrm{d}y)$$

is an **α-stable symmetric random field** such that

$$\langle X_s, f \rangle = \int_K f(x) \widetilde{X}_s(x) \mu(dx).$$

Sample paths smoothness

Baudoin, L. In progress, 2023

- Let $\alpha \in (0,2)$.
  - $\widetilde{X}_s$ is **unbounded a.s.** on the compact K when $s \leqslant d_h/d_w$.
  - If $s > d_h/d_w$, there exists a **modification** $\widetilde{X}_s^*$ of $\widetilde{X}_s$ such that

$$\lim_{\delta\to 0} \sup_{\substack{d(x,y)\leqslant\delta \\ x,y\in K}} \frac{\left|\widetilde{X}_s^*(x) - \widetilde{X}_s^*(y)\right|}{d(x,y)^{\min(s,1)d_w - d_h} \left|\ln d(x,y)\right|^{\beta}} < \infty.$$

  - Analoguous of Moving Average Fractional Stable Motion on $\mathbb{R}^n$.

A non exhaustive bibliography

1. F. Baudoin and L. Chen, Dirichlet fractional Gaussian fields on the Sierpiński gasket and their discrete graph approximations, Stochastic Processes and their Applications, 2023.
2. F. Baudoin and C. Lacaux, Fractional Gaussian fields on the Sierpiński gasket and related fractals, Journal d'Analyse Mathématique, 146, (2022), 719–739.
3. M. T.Barlow, Diffusions on fractals, in Lectures on Probability Theory and Statistics (Saint-Flour, 1995), Springer, Berlin, 1998, pp. 1–121.
4. M. T. Barlow and E. A. Perkins, Brownian motion on the Sierpiński gasket, Probab. Theory Related Fields 79 (1988), 543–623.
5. J. Kigami, Analysis on Fractals, Cambridge University Press, Cambridge, 2001.
6. A.N. Kolmogorov, Wienersche Spiralen und einige andere interessante Kurven in Hilbertsche Raum, C. R. (Dokl.) Acad. Sci. URSS 26 (1940) 115–118.
7. A. Lodhia, S. Sheffield, X. Sun and S. S. Watson, Fractional Gaussian fields: a survey, Probab. Surv. 13 (2016), 1–56.
8. B.B. Mandelbrot, J. Van Ness, Fractional Brownian motion, fractional noises and applications, SIAM Rev. 10 (1968) 422–437.

Thank you for your attention!

Here let $(K, \mathcal{A}, \mu)$ be a measurable space equipped with a σ-finite measure μ.

Normal contraction of a function

A function $v : K \to \mathbb{R}$ is called a **normal contraction of the function u** if for almost every $x, y \in K$,

$$|v(x) - v(y)| \leqslant |u(x) - u(y)| \text{ and } |v(x)| \leqslant |u(x)||.$$

Dirichlet form

Let $(\mathcal{E}, \mathcal{F})$, with $\mathcal{F} = \operatorname{dom} \mathcal{E}$, be a densely defined closed symmetric form on $L^2(K, \mu)$. The form $\mathcal{E}$ is a **Dirichlet form** if it is **Markovian,** i.e. if for any $u \in \mathcal{F}$ and any normal contraction v of u then $v \in \mathcal{F}$ and $\mathcal{E}(v, v) \leqslant \mathcal{E}(u, u)$.

Strongly local

The Dirichlet form $\mathcal{E}$ is **strongly local** if for any $f, g \in \mathcal{F}$ with compact support, when f is constant in a neighborhood of the support of g, then $\mathcal{E}(f, g) = 0$.

Regular Dirichlet form

The Dirichlet form $\mathcal{E}$ is **regular** if there exists $\mathcal{C} \subset \mathcal{C}_c(K) \cap \mathcal{F}$ which is dense in $\mathcal{C}_c(K)$ for the supremum norm and dense in $\mathcal{F}$ for the norm defined by

$$\|f\| = \left(\|f\|_{L^2} + \mathcal{E}(f, f)\right)^{1/2}.$$