A variational principle relating self-affine measures and self-affine sets

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Iterated function systems and their attractors

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- For every such IFS there exists a unique compact nonempty set X which satisfies $X = \bigcup_{i\in\mathcal{I}} T_i X$, and for every probability vector $(p_i)_{i\in\mathcal{I}}$ there exists a unique Borel probability measure m on $\mathbb{R}^d$ satisfying $m = \sum_{i\in\mathcal{I}} p_i (T_i)_* m$.

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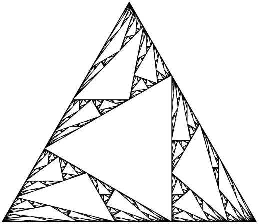
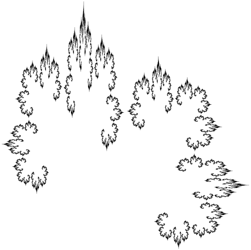
- General problem: find the Hausdorff dimensions of this *self-affine set* X and these *self-affine measures* m.

Some examples. . .





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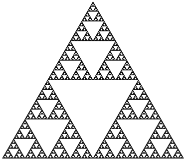

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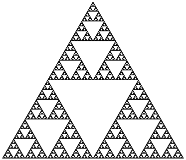

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The dimension s of the attractor solves $\sum_{i \in \mathcal{I}} r_i^s = 1$.

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- We can estimate $\dim_H X$ from above by using the sets $T_{i_1} \cdots T_{i_n} \overline{U}$ as covers.
- Since for every $n \geq 1$,

$$\sum_{i_1,\ldots,i_n \in \mathcal{I}} \left(\operatorname{diam} T_{i_1} \cdots T_{i_n} \overline{U}\right)^s = \sum_{i_1,\ldots,i_n \in \mathcal{I}} r_{i_1}^s \cdots r_{i_n}^s \left(\operatorname{diam} U\right)^s = \left(\operatorname{diam} U\right)^s,$$

we have $\dim_H X \leq s$.

- To get the lower bound we construct a measure m on $\mathbb{R}^d$ which assigns measure $r_{i_1}^s \cdots r_{i_n}^s$ to each of the sets $T_{i_1} \cdots T_{i_n} U$.

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- Let $\Sigma_{\mathcal{I}} = \mathcal{I}^{\mathbb{N}}$ be the set of all one-sided infinite sequences over $\mathcal{I}$, and let ν be the Bernoulli measure on $\Sigma_{\mathcal{I}}$ corresponding to the probability vector $(p_i)_{i \in \mathcal{I}}$.

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- Define a continuous function $\pi \colon \Sigma_{\mathcal{I}} \to \mathbb{R}^d$ by

$$\bigcap_{n=1}^{\infty} T_{i_1} \cdots T_{i_n} X = \{\pi[(i_k)_{k=1}^{\infty}]\}$$

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  which is a singleton set because the maps T_i are contractions.
- The measure $m := \pi_* \nu$ has Hausdorff dimension s and support X. The former is normally deduced by demonstrating the finiteness of the integral $\iint \|x - y\|^{-s} dm(x) dm(u)$.

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- Computing the dimensions of self-affine measures is also much harder and remains a wide open problem in dimension $d \geq 4$.
- If the linear parts A_i are algebraically degenerate (e.g. if they are all diagonal matrices) then various exceptional examples occur (e.g. the "carpet" fractals of Bedford and McMullen and their extensions by Gatzouras-Lalley, Das-Simmons, Feng-Wang, Fraser &c.).

- In 1988, Falconer obtained an upper bound for the dimension of a self-affine set essentially by "chopping" the sets $T_{i_1} \cdots T_{i_n} X$ into round pieces to create a more efficient cover. This gives a bound called the *affinity dimension* of $(T_i)_{i \in \mathcal{I}}$, written $\dim_{\mathrm{aff}}(T_i)_{i \in \mathcal{I}}$.

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- Feng more recently showed that the set of good translation vectors is also residual.
- *Explicit* examples of self-affine sets with known Hausdorff dimension remained rare until the late 2010s (e.g. Hueter-Lalley '95).

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- Given $B \in GL_d(\mathbb{R})$, let $\sigma_1(B) \geq \sigma_2(B) \cdots \geq \sigma_d(B)$ denote the singular values. For each $s \in [0, d]$ define

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- The affinity dimension s of $(T_i)_{i\in\mathcal{I}}$ is defined to be the unique $s \geq 0$ such that the quantity

$$P((T_i)_{i\in\mathcal{I}}; s) := \lim_{n\to\infty}\frac{1}{n}\log\sum_{i_1,\ldots,i_n\in\mathcal{I}}\varphi^s(A_{i_1}\cdots A_{i_n})$$

is equal to 0.

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- Subsequent research showed that $\dim_{\mathrm{H}} \pi_* \mu \leq \dim_{\mathrm{Lyap}} \mu$ and that there always exists an ergodic *equilibrium state* μ such that $\dim_{\mathrm{Lyap}} \mu = \dim_{\mathrm{aff}}(T_i)_{i\in\mathcal{I}}$.

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- A possible strategy for the general problem: understand enough about the equilibrium states μ, and the dimensions of measures of the form $\pi_*\mu$, to find a measure on the attractor X with Hausdorff dimension equal to $\dim_{\mathrm{aff}}(T_i)_{i\in\mathcal{I}}$.

Progress in the last decade: what is μ?

- In "typical" cases there is a unique equilibrium state μ, and in this case for every $i_1, \ldots, i_n \in \mathcal{I}$,

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- If ergodic, they are measurably isomorphic to finite extensions of Bernoulli processes.
- If $s = \dim_{\mathrm{aff}}(T_i)_{i \in \mathcal{I}}$ then there are not more than $\binom{d}{\lfloor s \rfloor}\binom{d}{\lceil s \rceil}$ distinct ergodic equilibrium states, or $\binom{d}{s}$ when s is an integer. In the integer case this bound is sharp. (Conjecture: the exact bound is $(d - \lfloor s \rfloor)\binom{d}{\lfloor s \rfloor}$ when s is non-integer.)

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- If we knew that the equilibrium states were Bernoulli measures then we could make the lower and upper bounds meet.
- However, this is *never* the case except when the maps A_i are conformal or admit an invariant linear subspace (M.-Sert '19).

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- Intuitive idea: for large n, many of the linear maps $A_{i_1} \cdots A_{i_n}$ look similar to one another (e.g. eigenspaces are in similar places, eigenvalues are of similar magnitude).
- If we "threw away" a subset of products corresponding to a set of small μ-measure, the ones which we keep would "almost commute". Would they (almost?) have a Bernoulli equilibrium state?

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- Clearly, $(T_{\mathtt{i}})_{\mathtt{i} \in \mathcal{I}^n}$ is an IFS with the same attractor as $(T_i)_{i \in \mathcal{I}}$. It also has the same affinity dimension.
- By taking n sufficiently large, can we find Bernoulli measures on $\Sigma_{\mathcal{I}^n}$ with Lyapunov dimension close to $\dim_{\mathrm{aff}}(T_i)_{i \in \mathcal{I}}$?

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- This is a problem since results on self-affine measures apply only to fully-supported Bernoulli measures.

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- We need a theorem showing that the desired *analytic* properties described above can be obtained in a way which ensures that $(T_{\mathtt{i}})_{\mathtt{i}\in\mathcal{J}}$ has the same *algebraic* features as $(T_i)_{i\in\mathcal{I}}$.

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This allowed deep results of Bárány, Hochman and Rapaport on planar self-affine *measures* to translate directly into results on planar self-affine *sets*.

Proof idea in the proximal & strongly irreducible case

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- Using a pigeonhole argument and non-atomicity of the distribution of the Oseledec subspaces, we can do this in a way which ensures that $\{A_{\mathtt{i}} \colon \mathtt{i} \in \mathcal{J}\}$ is an irreducible and dominated semigroup. Strong irreducibility follows.

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- Control on cardinality of $\mathcal{J}$ and on Lyapunov exponents implies control of the Lyapunov dimension of the measure of maximal entropy on $\Sigma_{\mathcal{J}}$.

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- We need to consider irreducibility and proximality across multiple representations (e.g. different exterior powers).
- There are very few subgroups of $\mathrm{GL}_2(\mathbb{R})$, resulting in what could be seen as a case-by-case argument depending on which linear algebraic group $(A_i)_{i\in\mathcal{I}}$ generates. In general dimensions no analogous case-by-case argument is possible.

...and now the result:

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Corollary

Let $(T_i)_{i\in\mathcal{I}}$ be a strongly irreducible affine iterated function system acting on $\mathbb{R}^3$ and satisfying the strong open set condition. Then the Hausdorff dimension of the attractor is equal to the affinity dimension of $(T_i)_{i\in\mathcal{I}}$.

An overview of the proof (with simplifications)

1. Choose an equilibrium state μ and use SAET and SMBT as before to find a set $\mathcal{J}_0 \subset \mathcal{I}^n$ of at least $e^{n(h(\mu)-\varepsilon)}$ words $\mathtt{i}$ such that the singular values of every $A_{\mathtt{i}}$ are $n\varepsilon$-close to the respective Lyapunov exponents.

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3. Extending those words by an *a priori* bounded amount, pass to a new set $\mathcal{J}_2 \subset \mathcal{I}^{n+k}$ of at least $e^{n(h(\mu)-3\varepsilon)}$ words which generate a narrow Schottky subsemigroup of the identity component and where the singular values are still $2n\varepsilon$-close to the respective Lyapunov exponents.

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4. Select some additional words $\mathtt{k}_1, \ldots, \mathtt{k}_t$ which, when appended to $\mathcal{J}_3$, ensure that a Zariski-dense subsemigroup of the identity component is generated. (Moreover, do this in such a way that substituting any power of $\mathtt{k}_i$ for the relevant word $\mathtt{k}_i$ has the same effect.)

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5. The set $\mathcal{J}_3 \cup \{\mathtt{k}_1, \ldots, \mathtt{k}_t\}$ no longer consists of words of a consistent length, so choose integers m, $r_1, \ldots, r_t$ such that

$$\mathcal{J}_4 = \{\mathtt{i}_1 \cdots \mathtt{i}_m \colon \mathtt{i}_j \in \mathcal{J}_3\} \cup \{\mathtt{k}_1^{r_1}, \ldots, \mathtt{k}_t^{r_t}\}$$

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6 Some of those words do not have *a priori* control on their singular values, so instead consider

$$\mathcal{J}_5 = \{\mathtt{i}_1 \cdots \mathtt{i}_{m+p} \colon \mathtt{i}_j \in \mathcal{J}_3\} \cup \{\mathtt{i}^p \mathtt{k}_1^{r_1}, \ldots, \mathtt{i}^p \mathtt{k}_t^{r_t}\}$$

where $\mathtt{i} \in \mathcal{J}_3$ is arbitrary, and p is large enough that $A_{\mathtt{i}}^p$ generates a Zariski-connected semigroup, and also large enough that the singular values are $3n\varepsilon$-controlled.

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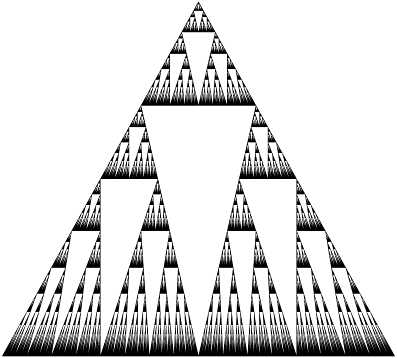
8. Control on the number of elements and their Lyapunov exponents implies control on the Lyapunov dimension.

A list of ingredients (not exhaustive):

- Bochi-Gourmelon: characterisations of domination
- Benoist: finding Zariski dense, narrow Schottky subsemigroups of semigroups of linear maps
- Tits: finding small generating sets for Zariski dense subsemigroups
- Abels-Margulis-Soifer: finding large proximal subsets of semigroups of linear maps
- Guivarc'h-Raugi: separating Lyapunov exponents for Bernoulli measures
- . . . and extensive prior results by many researchers on affine IFS.



Thanks for listening!