Random Fourier series vs. random wavelet series

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Multifractal analysis and self-similarity

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Some basic issues concerning wavelet expansions

- How “robust” are wavelet techniques ?
- How do wavelet series behave under the randomization of their wavelet coefficients ?
- How far does this behavior differ from random Fourier series ?
- Focus on Gaussian random wavelet series
- Understand ton this example he interplay between functional analysis and probability

A problem posed by Hilbert

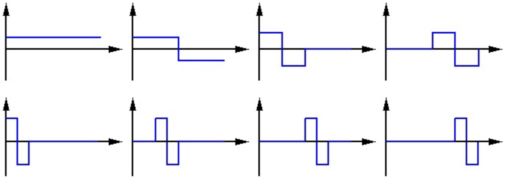
In 1873 Du Bois-Reymond proved that the Fourier series of a periodic continuous function may diverge at some points

Hilbert asked his PhD student Alfred Haar to determine if this drawback is inherent to any orthonormal basis, or if some bases can behave "better" than the trigonometric system

In 1909, Haar constructed the Haar basis to solve this problem

$$\begin{cases} \varphi(t) = 1 & \text{if } t \in (0,1), \\ \varphi(0) = \varphi(1) = 1/2, & \\ \varphi(t) = 0 & \text{else} \end{cases}$$

$$\psi(t) = \varphi(2t) - \varphi(2t-1)$$



credit : http ://fourier.eng.hmc.edu/e161/lectures/Haar/index.html

The $\begin{cases} \varphi(t) \\ 2^{j/2}\psi(2^j t - k), \ j \geq 0, \quad k = 0, \cdots 2^j - 1 \end{cases}$

form an orthonormal basis of $L^2([0,1])$

The Haar basis

Theorem : Let f be a continuous function on $[0, 1]$. Then the partial sums of the Haar expansion of f converge uniformly to f

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Theorem : (G. Bourdaud, 1995) The Haar basis is an unconditional basis of the Besov spaces $B_p^{s,p}(\mathbb{R})$ if and only if

$$\frac{1}{p} - 1 < s < \min\left(\frac{1}{p}, 1\right)$$

Haar-basis characterization of Besov spaces : Let $p > 0$

$$f \in B_p^{s,p} \quad \text{if} \quad \exists C, \forall j : \quad 2^{(sp-d)j} \sum_{j,k} |C_{j,k}|^p \leq C$$

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Wavelet bases have the same algorithmic structure as the Haar basis, but φ and ψ can be arbitrarily smooth

If wavelets are smooth enough, the function space characterizations hold without restriction on the range of indices

Unconditional bases supply robust expansions

Definition : Let X be a separable Banach space ; a sequence $(e_n)_{n\in\mathbb{N}}$ of elements of X is a Schauder basis if, for any $f \in X$, there exists a unique sequence of real numbers $(a_n)_{n\in\mathbb{N}}$ such that

$$\sum_{n\leq N} a_n e_n \to f \quad \text{in} \quad X$$

(for non separable spaces, strong convergence is replaced by weak-$*$ convergence)

It is an unconditional basis if

$$\exists C > 0, \quad \forall \varepsilon_n = \pm 1, \quad \left\| \sum_{n\in F} \varepsilon_n a_n e_n \right\|_X \leq C \left\| \sum_{n\in F} a_n e_n \right\|_X$$

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Wavelets bases are unconditional bases of “most” function spaces : L^p $(1 < p < \infty)$, C^α, Sobolev or Besov spaces

Wavelets vs. Fourier series

The trigonometric system is not an unconditional basis of "most" function spaces :
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Theorem (De Leeuw, Kahane and Katznelson, 1977) : Let $f \in L^2([0,1])$

$$f(t) = \sum_{n \in \mathbb{Z}} c_n \, e^{2i\pi nt}$$

Then, there exists a continuous function g such that

$$g(t) = \sum_{n \in \mathbb{Z}} d_n \, e^{2i\pi nt} \quad \text{and} \quad |d_n| \geq |c_n|$$

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Theorem : If $f(t) = \sum C_{j,k} \; \psi(2^j t - k)$ belongs to a Sobolev or Besov space for $1 < p, q < \infty$, then its Gaussian randomization

$$X_f(t) = \sum \chi_{j,k} C_{j,k} \; \psi(2^j t - k) \qquad (\; \chi_{j,k} : \text{IID standard Gaussians})$$

belongs to the same space

Random Fourier series vs. random wavelet series

Theorem : (Marcus and Pisier, 1981) Let $(a_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers and consider the two random Fourier series

$$\sum a_n R_n e^{int} \quad \text{and} \quad \sum a_n \chi_n e^{int}$$

where $(R_n)_{n\in\mathbb{Z}}$ is a sequence of independent Rademacher random variables and $(\chi_n)_{n\in\mathbb{Z}}$ is a sequence of independent standard Gaussian random variables. Then, almost surely, either both of them are continuous or both of them are not bounded.

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Theorem : Let $\chi_{j,k}$ be IID standard Gaussian RV. There exist wavelet series

$$f(t) = \sum C_{j,k}\psi(2^j t - k)$$

which are uniformly convergent and such that the randomization

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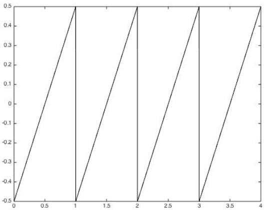
This result actually holds generically for almost every function f in the space of continuous functions (in the sense of prevalence)

Randomization of the sawtooth function

Let $\{t\}$ be the "sawtooth function"

$$\{t\} = t - [t] - \frac{1}{2}$$



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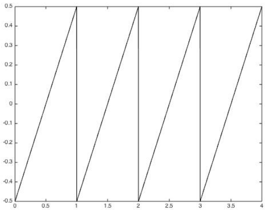


Its Fourier series is

$$\{t\} = -\sum_{m=1}^{\infty} \frac{\sin(2\pi mt)}{\pi m}$$

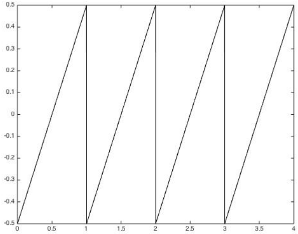
Wiener's Fourier expansion for Brownian motion on $[0, 1]$:

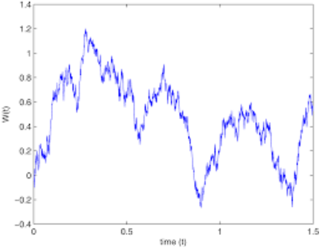
$$B(t) = \sqrt{2}\chi_0 t + \sum_{m=1}^{\infty} \chi_m \frac{\sin(2\pi mt)}{\pi m}$$

where (χ_m) is a sequence of I.I.D. standard Gaussian random variables

Therefore the Gaussian randomization of the sawtooth function is, up to a linear term, the Brownian motion

Explicit example of randomization





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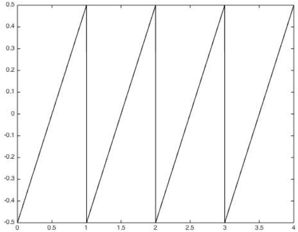
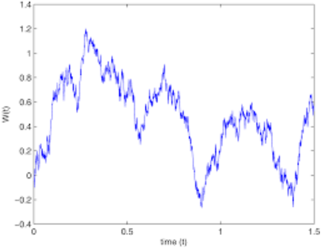
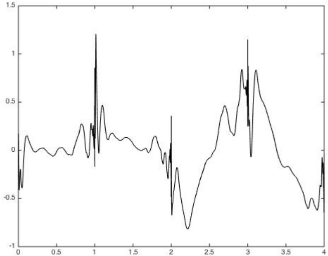




Theorem : The wavelet randomization is as smooth as the wavelet basis used, except at the discontinuities where it is a. s. unbounded



Pointwise regularity

Let f be a locally bounded function $\mathbb{R}^d \to \mathbb{R}$ and $t_0 \in \mathbb{R}^d$; $f \in C^\alpha(t_0)$ if there exist $C > 0$ and a polynomial P of degree less than α such that

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Gaussian randomization of wavelet series

Theorem : Let $\chi_{j,k}$ be IID standard Gaussian Random variables.

Let $f \in C^\varepsilon$ for an $\varepsilon > 0$

$$f(t) = \sum C_{j,k} \psi(2^j t - k)$$

Its Gaussian randomization is $X_f(t) = \sum C_{j,k} \chi_{j,k} \psi(2^j t - k)$

a. s. $\forall t$ $\quad h_{X_f}(t) \geq h_f(t)$

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Are there random sets of points where $h_{X_f}(t) \neq h_f(t)$?

Regularity of Gaussian processes

Most results concern almost sure regularity at a time t (Fernique, Marcus and Pisier, Talagrand, Ledoux, ...)

Example : Khintchine's law of the iterated logarithm for Brownian motion :

$$\forall t_0, \quad a.s. \qquad \limsup_{|\delta|\to 0} \frac{|B(t_0+\delta)-B(t_0)|}{\sqrt{|\delta|\log(\log(1/\delta))}} = 1.$$

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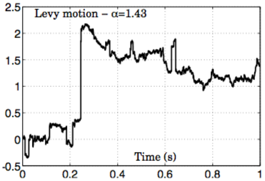

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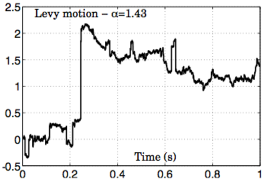

Pathwise pointwise modulus of continuity for Brownian motion

Slow points :

$$a.s. \qquad \limsup_{|\delta|\to 0} \frac{|B(t_0+\delta)-B(t_0)|}{\sqrt{|\delta|}} < \infty$$

Fast points :

$$a.s. \qquad \limsup_{|\delta|\to 0} \frac{|B(t_0+\delta)-B(t_0)|}{\sqrt{|\delta|\log(1/\delta)}} \geq C$$

Pointwise regularity of Gaussian processes

Can the Hölder exponent of a Gaussian process change between sample paths ?

If so, how big can the corresponding set be ?

Let X be a stochastic process such that there exists a deterministic function $\mathbf{H}_X(t)$ such that

$$\forall t, \quad \text{a.s.} \quad h_X(t) = \mathbf{H}_X(t).$$

The random Hölder set of X is

$$\mathbf{R}_X = \{t : \quad h_X(t) \neq \mathbf{H}_X(t)\}$$

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Antoine Ayache investigated new variants of Gaussian multifractional stochastic processes for which this set is non-empty with positive probability

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A general model investigated by Jean-Marie Aubry and S. J. in 2002

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Model 1 : The wavelet coefficients of the process at scale j are IID and can only take two values :

$$\mathbb{P}(C_{j,k} = 0) = 1 - 2^{(\eta-1)j} \quad \text{and} \quad \mathbb{P}(C_{j,k} = 2^{-\alpha j}) = 2^{(\eta-1)j}$$

Model 2 : The wavelet coefficients of the process at scale j are IID take the values : either 0, with probability $1 - 2^{(\eta-1)j}$
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Model 2 does not yield a Gaussian process. However its construction can be subordinated to the first one : Model 2 can be defined on $\Omega = \Omega_1 \times \Omega_2$ where Ω_1 is the probability space on which the first model is defined, thus yielding the nonvanishing coefficients $C^1_{j,k}$;
$C^2_{j,k} = C^1_{j,k}\chi_{j,k}$ where the $\chi_{j,k}$ I.I.D. Gaussian R.V.

For a given $\omega_1 \in \Omega_1$, $(\omega_2, t) \to X_2((\omega_1, \omega_2), t)$ is a Gaussian process

Multifractal spectra

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$$\begin{cases} \rho^1(H) & = \eta \quad \text{if } H = \alpha \\ & = -\infty \quad \text{else} \end{cases} \qquad \begin{cases} \rho^2(H) & = \alpha + \eta - H \quad \text{if } H \in [\alpha, \alpha + \eta] \\ & = -\infty \quad \text{else} \end{cases}$$

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The multifractal spectrum of f is : $\mathcal{D}_f(H) = \dim(\{x_0 : \quad h_f(x_0) = H\})$
where dim denotes the Hausdorff dimension

$$\forall f \in C^\varepsilon \text{ for an } \varepsilon > 0, \quad \forall H \geq 0, \qquad \mathcal{D}_X(H) \leq H \sup_{\beta \in (0,H]} \frac{\rho(\beta)}{\beta} \tag{1}$$

$$\text{Let} \qquad H_{max} = \min\left\{ H : \quad H \sup_{\beta \in (0,H]} \frac{\rho(\beta)}{\beta} = 1 \right\};$$

A wavelet series satisfies the *large deviation multifractal formalism* if its Hölder exponents are as small as allowed by (1), i.e. if equality holds in (1) for $H \leq H_{max}$ and $\mathcal{D}(H) = -\infty$ for $H > H_{max}$

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RWS satisfy the large deviation mutifractal formalism

The two processes X_1 and X_2 share the same multifractal spectrum. On any interval :

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Theorem : Let $X_1(t,\omega)$ be a generic sample path of the RWS X_1, and X_2 be its wavelet Gaussian randomization

- The random set $\mathbf{R}_{X_2}$, where the two exponents differ is an homogenous fractal set of Hausdorff dimension 1
- On any interval the random spectrum of X_2 is given by (2), so that X_2 is maximally random on any interval

Thank you

for your attention !