Multifractal analysis of weighted Birkhoff averages

Aihua FAN
(Picardie University, France)

Analyse Multifractale et Auto-similarité
CIRM, June 26-30, 2023

Plan

Study object: Topological dynamical system (X, T) and

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} w_n f(T^n x), \qquad (\overline{\lim_{N}} \frac{1}{N}\sum_{n=0}^{N-1} |w_n| > 0).$$

- Multifractal analysis of weighted Birkhoff averages
- Liapounov exponent of products of non-negative matrices

Problem of Multifractal analysis

What is the pointwise behavior of

$$\frac{1}{N}\sum_{n=0}^{N-1} w_n \varphi(T^n x)?$$

Set up

- (X, T): TDS; φ: real continuous function on X; (w_n): weight.
- Weighted sums: $S_n\varphi(x) = \sum_{k=0}^{n-1} w_k \varphi(T^k x)$.
- Level sets: $\alpha \in \mathbb{R}$

$$E(\alpha) = \{x \in X : n^{-1} S_n\varphi(x) \to \alpha\}.$$

- Multifractal spectrum

$$\alpha \mapsto \dim_H(E(\alpha)), \qquad \alpha \mapsto \mathrm{h_{top}}(E(\alpha)).$$

Problem How to compute $\dim_H(E(\alpha))$ or $\mathrm{h_{top}}(E(\alpha))$?

Background: Riesz products

Riesz measure on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is defined as weak*-limit:

$$\mu_a = \lim_{n\to\infty} \prod_{j=1}^{n} \left(1 + \ \mathrm{Re} a_j e^{2\pi i \lambda_j x}\right) dx$$

where $a_j \in \mathbb{C}$ with $|a_j| \le 1$ and $\lambda_j \in \mathbb{N}$ with $\lambda_{j+1}/\lambda_j \ge 3$.

1. (1989's) J.P. Kahane told me that J. Peyrière asked the question of multifractal analysis of Riesz products.
2. (Peyrière, 1975, $\frac{1}{2}$-thesis, Ann. Int. Fourier): For an interval $I_n(x)$ containing x with $|I_n(x)| = \frac{1}{\lambda_n}$

$$\log \mu_a(I_n(x)) = \sum_{j=1}^{n} \log \left(1 + \ \mathrm{Re} a_j e^{2\pi i \lambda_j x}\right) - \log \lambda_n + o(\log \lambda_n).$$

3. (Fan, 1993 Bull. Sci. Math.): Riesz products were used to analyze a given Riesz product, incompletely.
4. If $a_k = a$ and $\lambda_k = 3^k$, μ_a is T_3-invariant, ergodic (in fact, a Gibbs measure), where $T_3 x = 3x \mod 1$.
5. (Fan, J. Stat. Phys. 1997) Multifractal analysis of

$$\log \prod_{k=1}^{n} g_k(\lambda_k x) = \sum_{k=1}^{n} f_k(\lambda_k x).$$

Background (continued): dimensions of measures

Let μ be a (probability) Borel measure on a metric space.
Lower and upper Hausdorff dimensions of μ are defined by:

$$\dim_* \mu := \inf\{\dim E : \mu(E) > 0\}, \qquad \dim^* \mu := \inf\{\dim E : \mu(E^c) = 0\}.$$

1. Peyrière (1975) was the first to estimate $\mathbf{dim}_* \mu$ and $\mathbf{dim}^* \mu$ (for μ Riesz product). He didn't talk about "dimensions", but the fact. The same for his result about the measure concerning Mandelbrot cascades (1976).
2. Fan (1989 Thesis, 1994 Studia Math.) defined $\mathbf{dim}_* \mu$ and $\mathbf{dim}^* \mu$ and proved

$$\mathbf{dim}_* \mu = \text{ess inf}_\mu \underline{D}(\mu, x), \qquad \mathbf{dim}^* \mu = \text{ess sup}_\mu \underline{D}(\mu, x)$$

   where

$$\underline{D}(\mu, x) = \varliminf_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}.$$

3. Tamashiro (1995), Heurteaux (1998): Packing dimensions $\text{Dim}_* \mu$ and $\text{Dim}^* \mu$ are similarly defined and shares the similar equalities using $\overline{D}(\mu, x)$.
4. In nice cases, $D(\mu, x)$ exists μ-a.e. (Young 1982; Ledrappier-Young 1985, Barreira-Pesin-Schmelin 1995, $\cdots$)
5. Multifractal analysis of a measure μ is a "global" study of the local behavior of $\mu(B(x, r))$ as $r \to \mathbf{0}$, a natural extension of the study on "dimensions" of μ.

Weights (I): Davenport weights

Def. The Davenport exponent $D((w_n))$ of (w_n) is the best $h > 0$:

$$\sup_{t\in[0,1)} \left| \sum_{n=0}^{N-1} w_n e^{2\pi int} \right| = O_h(N/\log^h N).$$

Theorem (Fan 2017, ETDS)

If $(w_n) \in \ell^\infty$ with $D((w_n)) > \frac{1}{2}$, then for any MPDS $(X, \mathcal{B}, \nu, T)$ and $f \in L^1(\nu)$, we have ν-a.e.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} w_n f(T^n x) = 0.$$

1. Davenport: $D((\mu(n))) = +\infty$.
2. Invariant measures are not useful for analyzing wighted ergodic averages.

Weigts (II): Gelfond weights

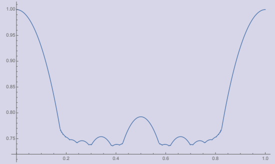
Def. The Gelfond exponent $G((w_n))$ of (w_n) is the best $0 < d < 1$:

$$\sup_{t\in[0,1)} \left| \sum_{n=0}^{N-1} w_n e^{2\pi int} \right| = O_d(N^d).$$

Theorem (Fan-Schmeling-Shen 2021, DCDS)

We can compute the exact value $G((w_n))$ for generalized Thue-Morse sequences

$$w_n = t_n^{(c)} = e^{2\pi i c s_2(n)}.$$



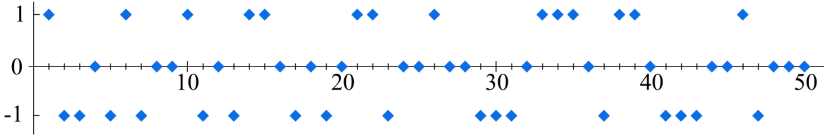
$c \mapsto D(t_n^{(c)})$

1. Gelfond (1968): $G((t_n^{(1/2)})) = \frac{\log 3}{\log 4}$.
2. FSS (2021): $\frac{\log(1-2\cos 2\pi c)}{\log 4}$ if $|c - 1/2| < 0.0718$; $1 + \frac{\cos c\pi}{\log 2}$ if $|c| < 0.175$ (two Sturmian measures).
3. Mauduit-Sarkozy (2018): estimation. Fouvry-Mauduit, Mauduit-Rivat, $\cdots$

Weights (III): Möbius function

The Möbius function $\mu : \mathbb{N} \to \mathbb{N}$ is a multiplicative arithmetic function defined by

$$\mu(1) = 1,\ \mu(p) = -1 \text{ and } \mu(p^k) = 0 \text{ if } k \geq 2$$



Sarnak's conjecture. $h_{\text{top}}(X, T) = 0 \Longrightarrow \mu \perp (X, T)$, i.e.

$$\forall f \in C(X), \forall x \in X, \quad \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(n) f(T^n x) = 0.$$

1. Not multifractal: $\frac{1}{N}\sum_{n=0}^{N-1} \mu(n) f(T^n x)$.
2. $(\mu(n))$ is a "difficult" weight.

Works done

- Fan Aihua, Multifractal analysis of infinite products. J. Stat. Phys. (1997).
- Fan Aihua, Multifractal analysis of weighted ergodic averages. Adv. Math. (2021).
- Bárány Balázs, Rams Michal, Shi Ruxi, On the multifractal spectrum of weighted Birkhoff averages. Discrete Contin. Dyn. Syst. 42 (2022). Stationary weights.
- Bárány Balázs, Rams Michal, Shi Ruxi, Spectrum of weighted Birkhoff average. Studia Math. (2023). $w_n = \frac{1}{n}$.
- Fan Aihua, Wu Meng, A topological version of a Fursternberg -Kesten theorem, preprint. Question asked after Wu Meng's thesis, 2013. Matrices selected by Thue-Morse..
- Fan Aihua, Verbitskiy Evgeny, Computation of Lyapounov exponent of some products of non-negative matrices, preprint.
- Fan Aihua, Multifractal analysis of weighted ergodic averages on Markovian systems, in preparation.

Classical situation: $\forall n, w_n = 1$.

Plenty of works done. Some authors:

Continuous potentials

1. Fan-Feng (1998), Fan-Feng-Wu (2001): Conditional variation principle for SFT.
2. Takens-Verbitsiiy (2003): systems of specifications.
3. Fan-Liao-Peyriere (2008): systems of specifications, φ Banach-valued.
4. E. Olivier (1999)
5. ...

Hölder potentials

1. Schmeling (1999)
2. Barreira-Saussol-Schmeling (2002)
3. ...

Thermodynamic formalism on $\{0, 1, \cdots, q-1\}^{\mathbb{N}}$ (Fan, 1997)

Pressure function (definition and hypothesis):

$$\Psi(\beta) := \lim_{n\to\infty} \frac{1}{n} \log_q \mathbb{E}_{Bernoulli} \exp(\beta S_n \varphi) + 1$$

Fundamental inequalities (key fact and hypothesis): $\exists C > 0$ such that

$$\forall \ell < m < n, \quad C^{-1} \le \frac{Z_{\ell,n}(\beta)}{Z_{\ell,m}(\beta) Z_{m,n}(\beta)} \le C$$

where

$$Z_{m,n}(\beta) := \mathbb{E}_{Bernoulli} \exp(\beta[S_n\varphi - S_m\varphi]), \quad m < n.$$

Gibbs measures ν_β: Weak limit of

$$Z_{0,n}(\beta)^{-1} \exp(\beta S_n \varphi(x))\, dx.$$

Large deviation of Ellis is applicable: the free energy function is well defined for Gibbs measures.

1. Same holds for SFTs: Parry measure replaces the Bernoulli measure.
2. Markov dynamics: conformal measure replaces the Bernoulli measure.

Theorem (Fan, 1997, 2021)

Suppose
(1) φ is Hölder, (2) $\Psi(\beta)$ is well defined, (3) Ψ is differentiable.
Then for $\alpha = \Psi'(\beta)$ we have

$$\dim_H E(\alpha) = \dim_P E(\alpha) = \Psi^*(\alpha) := \inf_x (\alpha x - \Psi(x)) = \alpha\beta - \Psi(\beta).$$

1. If (w_n) is generic for some invariant measure, Φ is well defined (Fan-Wu).
2. If, furthermore, φ depends on finite coordinates, Ψ is analytic (Ruelle).
3. If Ψ is not differentiable, the sub-derivative $\partial\Psi$ plays the role.
4. (Fan, 1997): study of $\sum_{n=1}^{N} g_n(\lambda_n x)$ under $\lambda_n | \lambda_{n+1}$ ($\longrightarrow$ fundamental inequalities).

Computation of spectrum (Liapounov exponent)

Assumption 1: $\varphi(x) = \varphi(x_0, x_1)$.
Assumption 2: $w_n \in \{v_1, v_2\}$ (or a finite set). Let

$$A_w(\beta) := (e^{w\beta\varphi(x_0,x_1)}) \qquad (q \times q \text{ matrix})$$

Observation: $\Psi(\beta) = \lim_{n\to\infty} \frac{1}{n} \log \|A_{w_0}(\beta)A_{w_1}(\beta)\cdots A_{w_n}(\beta)\|$.

Theorem (Fan-Wu, 2022)

The Lyapounov exponent $\Psi(\beta)$ is well defined if (w_n) is generic for some shift invariant measure.

1. All primitive substitutive weights are good (Examples: Thue-Morse, Fibonacci).
2. $(\mu(n)^2)$ is good (i.e. generic for some ergodic measure).
3. $(\mu(n))$ is a big problem.

Liapounov exponent: example of square-free weight $\mu(n)^2$

(1) Question (form 1): Multifractal analysis of

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu(n)^2 x_n x_{n+1}, \quad (x \in \{0,1\}^{\mathbb{N}}.$$

(2) Question (form 2): Liapounov exponent of $(A_{\mu(n)^2}(\beta))$:

$$A_0(\beta) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad A_1(\beta) = \begin{pmatrix} 1 & 1 \\ 1 & e^{\beta} \end{pmatrix}.$$

Ingredients

- $(\mu(n)^2)$ is generic for Mirsky measure (Mirsky[1949], Wu[1993], Sarnak[2012], Cellarosi-Sinai[2013], Peckner[2015], ALR[2015], $\cdots$).
- Riemann-type function:

$$\zeta_a(s) = \sum_{n=1}^{\infty} \frac{a^{\Omega(n)}}{n^s} = \prod \left(1 - \frac{a}{p^s}\right)^{-1},$$

where $\Omega(n)$ is the number of primes with multiplicity of n.

Liapounov exponent: example of square-free (continued)

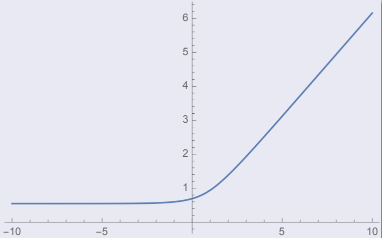
Theorem (Fan-Verbitskiy, 2022)

$$\begin{aligned}\Psi(\beta) &= (1-2F_1-3F_{11}-4F_{111})\log 2 + \\ &+F_1\log||A_1(\beta)|| + F_{11}\log||A_1(\beta)^2|| + F_{111}\log||A_1(\beta)^3||,\end{aligned}$$

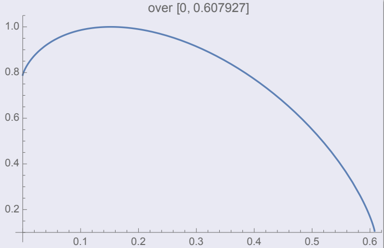
where

$$\begin{aligned}F_1 = F_{10} &= \frac{1}{\zeta_1(2)} - \frac{2}{\zeta_2(2)} + \frac{1}{\zeta_3(2)} = 0.0881459; \\ F_{11} = F_{110} &= \frac{1}{\zeta_2(2)} - \frac{2}{\zeta_3(2)} = 0.0716601; \\ F_{111} = F_{1110} &= \frac{1}{\zeta_3(2)} = 0.125487.\end{aligned}$$

Liapounov exponent: Square-free (graphs)



$\Psi(\beta)$



$\Psi^*(\alpha)$

Generalization to $\mathcal{B}$-free set $F_{\mathcal{B}}$

(P. Erdös) Let $\mathcal{B} = \{b_k : k \geq 1\} \subset \{2, 3, 4, \cdots\}$,

$$b_k \uparrow +\infty, \quad (b_k, b_{k'}) = 1, \quad \sum_{k=1}^{\infty} \frac{1}{b_k} < \infty.$$

An integer $n \in \mathbb{N}^*$ is $\mathcal{B}$-free if $b_k \not| n$ for all $k \geq 1$.

Theorem (El Abdalaoui, Lemanczyk, de la Rue, 2015)

(1) $1_{F_{\mathcal{B}}} \in \{0, 1\}^{\mathbb{N}}$ generates a subshift of entropy $\prod(1 - 1/b_k)$.

(2) Mirsky measure $\mu_{\mathcal{B}}$ is ergodic and of zero entropy.

(3) $1_{F_{\mathcal{B}}}$ is $\mu_{\mathcal{B}}$-generic.

A topological version of Furstenberg-Kesten theorem

Theorem (Furstenberg-Kesten, 1960)

Let $A : X \to \mathrm{GL}(d, R)$ with $\log^+ \|A\| \in L^1(\mu)$ for some T-invariant ergodic measure μ. Then for μ-a.e. ω, the following limit exists

$$L(\omega) := \lim \frac{1}{n} \log \|A(\omega)A(T\omega) \cdots A(T^{n-1}\omega)\|.$$

Question. Does the limit exists for a μ-generic point ?

NO, in general. Even for uniquely ergodic systems (M. Herman 1981, P. Walters 1986).

Products of non-negative matrices

Set up

- $\Omega := \{0, 1, \cdots, m-1\}^{\mathbb{N}}$; $\sigma : \Omega \to \Omega$ shift map.
- $A(\cdot)$: continuous function taking non-negative matrices as values.
- $\forall \omega$, $A(\omega)$ is allowable (no row, no column is zero vector)

Theorem (Fan-Wu, 2022)

Suppose ν is a σ-invariant and ergodic measure (H0) and

(H1) (Primitivity) there exist $x \in \text{supp}(\nu)$ and $\ell_0 \geq 1$ such that

$$A(x)A(\sigma x) \cdots A(\sigma^{\ell_0 - 1} x) > 0;$$

(H2) (Non-nullity) Non zero entries of $A(\omega)$ are uniformly bounded from below:

$$\min\{A(\omega)_{i,j} : A(\omega)_{i,j} \neq 0, \omega \in \Omega\} > 0.$$

(H3) (Genericity) ω is ν-generic

Then the Lyapunov exponent $L(\omega)$ is well defined.

How about $\mu(n)$?

Open problem 1 What is the multifractal spectrum of

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n) x_n x_{n+1}, \quad (x \in \{0,1\}^{\mathbb{N}})?$$

Open problem 2 Is the Liapounov exponent of $(A_{\mu(n)}(\beta))$ well defined where

$$A_0(\beta) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad A_1(\beta) = \begin{pmatrix} 1 & 1 \\ 1 & e^{\beta} \end{pmatrix}, \quad A_{-1}(\beta) = \begin{pmatrix} 1 & 1 \\ 1 & e^{-\beta} \end{pmatrix}?$$

Open problem 3 What are the frequencies of patterns

$$ab, abc \qquad (a, b, c \in \{-1, 1\})$$

in $(\mu(n))$?

Thank you for your attention !