Uniform approximation via continued fractions

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Diophantine approximation

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For any $x \in \mathbb{R}$*, there exist infinitely many pairs* $(p, q) \in \mathbb{Z} \times \mathbb{N}$ *such that*

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2}.$$

Diophantine approximation

Theorem (Dirichlet, 1842)

Given $x \in \mathbb{R}$ and $t > 1$, there exists pair $(p, q) \in \mathbb{Z} \times \mathbb{N}$ such that

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{qt} \quad \text{and} \quad 1 \leq q < t.$$

Corollary

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Uniform vs asymptotic Diophantine approximation

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For any $x \in \mathbb{R}$*, there exist infinitely many pairs* $(p, q) \in \mathbb{Z} \times \mathbb{N}$ *such that*

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Most of the Metric Theory of Diophantine Approximation answers the following:

Question

What happens if the RHS of (2) is replaced by a faster decreasing function of q? (Khinitchine-type theorems)

Uniform vs asymptotic Diophantine approximation

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Most of the Metric Theory of Diophantine Approximation answers the following:

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What happens if the RHS of (2) is replaced by a faster decreasing function of q? (Khinitchine-type theorems)

Alternatively

Can try to replace the RHS of (1) by a faster decreasing function of t?(Improving Dirichlet's theorem)

Improvements to Dirichlet's corollary

Let $\psi : [t_0, \infty) \to \mathbb{R}_+$ be a non-increasing function with $t_0 \geq 1$ fixed.

$$W(\psi) = \left\{ x \in [0,1) : \left| x - \frac{p}{q} \right| < \psi(q) \quad \text{for i.m. } (p,q) \in \mathbb{Z} \times \mathbb{N} \right\}.$$

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Jarník-Besicovitch set: For any $\tau \geq 2$,

$$J(\tau) = \left\{ x \in [0,1) : \left| x - \frac{p}{q} \right| < \frac{1}{q^\tau} \quad \text{for i.m. } (p,q) \in \mathbb{Z} \times \mathbb{N} \right\}.$$

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Theorem (Khintchine, 1924)

$$\lambda(J(\tau)) = \begin{cases} 0 & \text{if} \quad \sum_{q=1}^{\infty} q^{1-\tau} < \infty \\ 1 & \text{if} \quad \sum_{q=1}^{\infty} q^{1-\tau} = \infty. \end{cases}$$

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Theorem (Jarník–Besicovitch, 1929)

$$\dim_{\mathrm{H}} J(\tau) = \frac{2}{\tau}.$$

$J(\tau)$ in terms of entries of continued fraction

Every irrational number $x \in [0, 1)$ has a *continued fraction expansion*,

$$x = [a_1(x), a_2(x), \dots]; \quad a_i(x) \in \mathbb{Z}^+ \text{ for each } i \in \mathbb{N}.$$

In particular,

$a_1(x) = \lfloor \frac{1}{x} \rfloor$ and $a_n(x) = \lfloor \frac{1}{T^{n-1}(x)} \rfloor$ for $n \geq 2$, where the Gauss transformation $T : [0, 1) \to [0, 1)$ is defined as

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$$\frac{p_n(x)}{q_n(x)} = [a_1(x), ..., a_n(x)]$$

(p_n, q_n coprime) are called the n-th *convergents* of x.

$$p_{n+1} = a_{n+1}(x)p_n + p_{n-1}, \quad q_{n+1} = a_{n+1}(x)q_n + q_{n-1}, \quad n \geq 0.$$

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- *Legendre's theorem: if* $\left| x - \frac{p}{q} \right| < \frac{1}{2q^2}$ *then* $\frac{p}{q} = \frac{p_n(x)}{q_n(x)}$, *for some* $n \geq 1$.

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- *Speed of approximation:*

$$\frac{1}{(a_{n+1}(x)+2)q_n^2(x)} \leq \left|x - \frac{p_n(x)}{q_n(x)}\right| \leq \frac{1}{a_{n+1}(x)q_n^2(x)}.$$

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$$q_n^2(x) \asymp e^{\log|T'(x)| + \cdots + \log|T'(T^{n-1}(x))|}$$

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- *For* $\tau \geq 2$

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$$J(\tau) = \left\{x \in [0,1) : a_n(x) \geq e^{((\tau-2)/2)(\log|T'(x)|+\cdots+\log|T'(T^{n-1}(x))|)} \text{ for i.m. } n \in \mathbb{N}\right\}.$$

Improving Dirichlet's theorem

Definition (Dirichlet improvable set)

Let $\psi : [t_0, \infty) \to \mathbb{R}_+$ *be a non-increasing function with* $t_0 \geq 1$ *fixed.*
A real number x *is called* ψ*-**Dirichlet improvable** if the system*

$$|qx - p| < \psi(t), \quad \text{and} \quad |q| < t$$

has a non trivial integer solution for all t *large enough.*
The set of ψ*-Dirichlet improvable numbers is denoted by* $D(\psi)$.

Elements of the complementary set, $D(\psi)^c$, *will be referred to as* ψ*-Dirichlet non-improvable numbers.*

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Question

- *Can one characterize $x \in D(\psi)$ in terms of its continued fraction expansion?*
- *What is the necessary and sufficient condition for non-increasing ψ so that $D(\psi)$ has zero–full measure?*

Criteria for Dirichlet's improvability

Some observations (Kleinbock–Wadleigh, 2018)

Let ψ *be non-increasing and suppose* $t\psi(t) < 1$ *for all large* t*. Then*

$$x \in D(\psi) \iff |q_{n-1}x - p_{n-1}| < \psi(q_n) \text{ for all } n \gg 1.$$

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$$\text{Cassels, 1971}: (1+[a_{n+1}, a_{n+2}, \dots][a_n, a_{n-1}, \dots, a_1])^{-1} = q_n|q_{n-1}x - p_{n-1}|$$

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Lemma (Kleinbock–Wadleigh, 2018)

Let $x \in [0,1) \smallsetminus \mathbb{Q}$*, and let* $\psi : [t_0, \infty) \to \mathbb{R}_+$ *be non-increasing function with* $t\psi(t) < 1$ *for all* $t \geq t_0$ *and* $\Psi(t) = \frac{t\psi(t)}{1-t\psi(t)}$*. Then*

(i) $x \in D(\psi)$ *if* $a_{n+1}(x)a_n(x) \leq \frac{1}{4}\Psi(q_n)$ *for all sufficiently large* n*.*

(ii) $x \in D(\psi)^c$ *if* $a_{n+1}(x)a_n(x) > \Psi(q_n)$ *for infinitely many* n*.*

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$$\left\{x \in [0,1) : a_n(x)a_{n+1}(x) \geq \Psi(q_n) \text{ for infinitely many } n \in \mathbb{N}\right\} \subset D(\psi)^c.$$

The Metrical theory of continued fractions

Let $\Phi : \mathbb{N} \to \mathbb{R}_{\geq 1}$ be a positive function with $\lim_{n\to\infty} \Phi(n) = \infty$ and for $m \in \mathbb{N}$, define

$$\mathcal{F}_m(\Phi) := \{x \in [0,1) : a_n(x) \cdots a_{n+m-1}(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N}\}.$$

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Theorem (Borel–Bernstein, 1912)

Then

$$\lambda(\mathcal{F}_1(\Phi)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \frac{1}{\Phi(n)} < \infty, \\ 1 & \text{if } \sum_{n=1}^{\infty} \frac{1}{\Phi(n)} = \infty. \end{cases}$$

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$$\lambda(\mathcal{F}_2(\Phi)) = \begin{cases} 0 & \text{if} \quad \sum_{n=1}^{\infty} \frac{\log \Phi(n)}{\Phi(n)} < \infty \\ 1 & \text{if} \quad \sum_{n=1}^{\infty} \frac{\log \Phi(n)}{\Phi(n)} = \infty. \end{cases}$$

Main Theorem

We consider a weighted generalization of $\mathcal{F}_m(\Phi)$: take $\mathbf{t} = (t_0, \ldots, t_{m-1}) \in \mathbb{R}_+^m$ and $\Phi : \mathbb{N} \to \mathbb{R}_{\geq 1}$, and define

$$\mathcal{F}_{\mathbf{t},m}(\Phi) := \left\{ x \in [0,1) : \prod_{i=0}^{m-1} a_{n+i}^{t_i}(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N} \right\}.$$

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We prove the following dichotomy statement for the Lebesgue measure of $\mathcal{F}_{\mathbf{t},m}(\Phi)$.

Theorem (B.–Hussain–Kleinbock–Wang, 2023)

Let $\Phi : \mathbb{N} \to \mathbb{R}_{\geq 1}$*. Then*

$$\lambda\big(\mathcal{F}_{\mathbf{t},m}(\Phi)\big) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \frac{(\log \Phi(n))^{\ell-1}}{\Phi(n)^{1/t_{\max}}} < \infty, \\ 1 & \text{if } \sum_{n=1}^{\infty} \frac{(\log \Phi(n))^{\ell-1}}{\Phi(n)^{1/t_{\max}}} = \infty, \end{cases}$$

where

$$t_{\max} = \max\{t_i : 0 \leq i \leq m-1\}, \ \ell = \#\{i : t_i = t_{\max}\}.$$

Hausdorff dimension for $B = 1$ and $B = \infty$

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$$\log B = \liminf_{n \to \infty} \frac{\log \Phi(n)}{n}.$$

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$$\log B = \liminf_{n \to \infty} \frac{\log \Phi(n)}{n}.$$

(1) *If* $B = 1$, *then* $\dim_{\mathrm{H}} \mathcal{F}_{\mathbf{t},m}(\Phi) = 1$.

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(1) *If* $B = 1$, *then* $\dim_{\mathrm{H}} \mathcal{F}_{\mathbf{t},m}(\Phi) = 1$.

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$$\dim_{\mathrm{H}} \mathcal{F}_{\mathbf{t},m}(\Phi) = \frac{1}{1+b}.$$

Hausdorff dimension for $B = 1$ and $B = \infty$

$$\mathcal{F}_{\mathbf{t},m}(\Phi) := \left\{ x \in [0,1) : \prod_{i=0}^{m-1} a_{n+i}^{t_i}(x) \geq \Phi(n) \text{ for infinitely many } n \in \mathbb{N} \right\}.$$

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(3) *If* $1 < B < \infty$*?*

Hausdorff dimension when $1 < B < \infty$

Theorem (B.–Hussain–Kleinbock–Wang, 2023)

Let $\Phi : \mathbb{N} \to \mathbb{R}_{\geq 1}$ *be such that* $1 < B < \infty$*, and let* $\mathbf{t} = (t_0, t_1) \in \mathbb{R}^2_+$*. Then*

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where

$$f_{t_0,t_1}(s) := \frac{s^2}{t_0 t_1 \cdot \max\left\{ \frac{s}{t_1} + \frac{1-s}{t_0}, \frac{s}{t_0} \right\}}.$$

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Some observations:

- Order of exponent. For instance,

$$f_{2,1}(s) = \frac{s^2}{1+s}, \text{ and } f_{1,2}(s) = \begin{cases} \frac{s^2}{2-s} & \text{if } s \leq \frac{2}{3}; \\ \frac{s}{2} & \text{if } s > \frac{2}{3}. \end{cases}$$

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  $f_{2,1}(s) < f_{1,2}(s)$ for any $1/2 < s < 1$.
- $\dim_{\mathrm{H}} \mathcal{F}_{(2,1)}(\Phi) > \dim_{\mathrm{H}} \mathcal{F}_{(1,2)}(\Phi)$.

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The sketch of proof is divided into three parts.

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1. First, calculate the upper bound for the Hausdorff dimension
   - for the posited dimensions s, show $\sum_{i=1}^{\infty} |U_i|^s < \infty$ for specific coverings $\{U_i\}_{i \geq 1}$ of a set say X.
2. Second, calculate the lower bound for the Hausdorff dimension
   1. a Cantor-type construction,
   2. the mass distribution principle
3. Use a limiting process to show these are the same.

Thank you!