Complex Generalized Integral Means Spectrum for Drifted Whole-Plane SLE

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Multifractal Analysis and Self-Similarity
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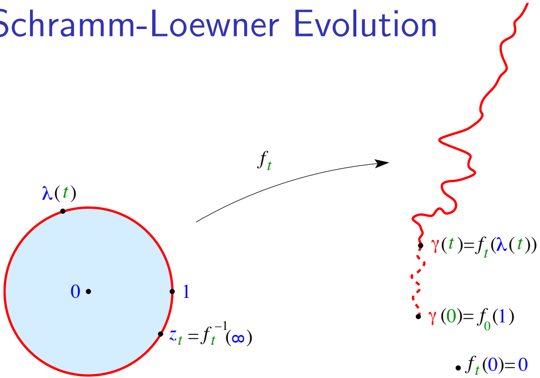
Whole-Plane Schramm-Loewner Evolution

- 

- $$z \in \mathbb{D}, \ \ \frac{\partial}{\partial t} f_t(z) = z \frac{\partial}{\partial z} f_t(z) \frac{\lambda(t) + z}{\lambda(t) - z}, \ \ \lambda(t) = \exp(i\sqrt{\kappa} B_t + iat)$$

  $$f_t(e^{-t}z) \to z, \ t \to +\infty; \ \ \kappa = 0, \ a = 0: \ \ f_t(z) = \frac{e^t z}{(1-z)^2} \ \ \text{(Koebe)}$$

- $1/f(1/z)$ is the **bounded exterior version** from $\mathbb{C} \setminus \overline{\mathbb{D}}$ to the slit plane [Beliaev & Smirnov '09].

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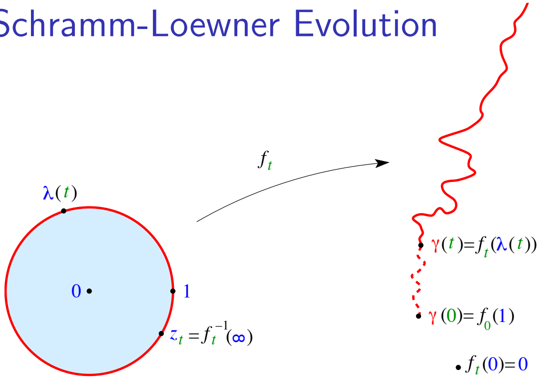

- $$z \in \mathbb{D}, \quad \frac{\partial}{\partial t} f_t(z) = z \frac{\partial}{\partial z} f_t(z) \frac{\lambda(t)+z}{\lambda(t)-z}, \quad \lambda(t) = \exp(i\sqrt{\kappa} B_t + iat)$$

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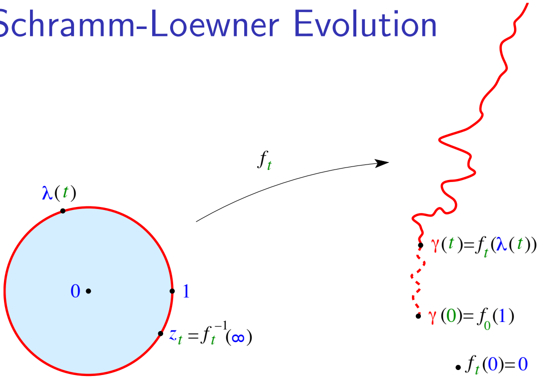

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Logarithmic Spiral

$\kappa = 0$, $a \neq 0$

$$\gamma(t) = \exp\left[(1 + ia)t\right], \quad t \geq 0.$$

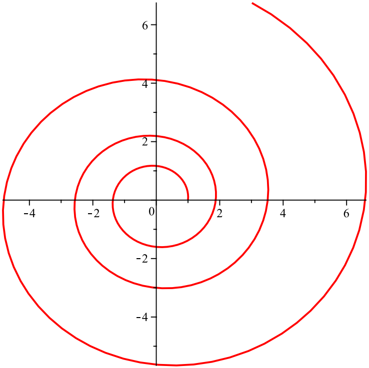

Figure: Logarithmic spiral $\gamma(t)$ for $a = 5$, restricted to $t \geq 0$.

Complex Generalized Integral Means Spectrum

- Consider a (**random**) Riemann map Φ on $\mathbb{D}$
- For $(p,q) \in \mathbb{C}^2$, define the complex generalized integral means

$$\mathcal{I}(r,p,q,\Phi) := \int_0^{2\pi} \left| \frac{\Phi'(re^{i\theta})^p}{\Phi(re^{i\theta})^q} \right| d\theta,\ 0 < r < 1$$

- **Expected:** $\mathbb{E}\,\mathcal{I}(r,p,q,\Phi) := \int_0^{2\pi} \mathbb{E} \left| \frac{\Phi'(re^{i\theta})^p}{\Phi(re^{i\theta})^q} \right| d\theta,\ 0 < r < 1.$
- Define

$$\beta_\Phi(p,q) := \limsup_{r\to 1^-} \frac{\log(\mathcal{I}(r,p,q,\Phi))}{\log(\frac{1}{1-r})}$$

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$$\mathcal{I}(r,p,q,\Phi) \overset{r\to 1^-}{\asymp} \frac{1}{(1-r)^{\beta_\Phi(p,q)}}.$$

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Integral means spectrum & harmonic measure

- The standard **integral means spectrum** $(p \in \mathbb{R}, q = 0)$ is related to the **multifractal spectrum** of the **harmonic measure** ω on the boundary of the image domain.
- For $\alpha \geq 1/2$, $\mathcal{E}_\alpha$ set of points z on the boundary where

$$\omega(B(z,r)) \sim r^{\alpha}, \quad r \to 0.$$

- Multifractal spectrum: $f(\alpha) = D_{\mathrm{Hausdorff}}(\mathcal{E}_\alpha)$.
- Integral means spectrum β to f by **Legendre transform**,

$$\frac{1}{\alpha} f(\alpha) = \inf_{p \in \mathbb{R}} \left\{ \beta(p) - p + 1 + \frac{1}{\alpha} p \right\},$$

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- Harmonic measure and logarithmic spiral **mixed** multifractal spectrum for $p \in \mathbb{C}$ [Binder '98]

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Generalized Integral Means Spectrum ($q \neq 0$)

- Unified treatment of the bounded and the unbounded cases.
- $f(z)$ **unbounded interior whole-plane SLE**
- $f^{[-1]}(z) := 1/f(1/z)$ the **bounded exterior whole-plane SLE** of Beliaev & Smirnov

$$|(f^{[-1]})'(z)^p| = |z^{-2p}| \left| \frac{f'(1/z)^p}{f(1/z)^{2p}} \right|, \quad q = 2p$$

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Generalized Spectrum for the Logarithmic Spiral

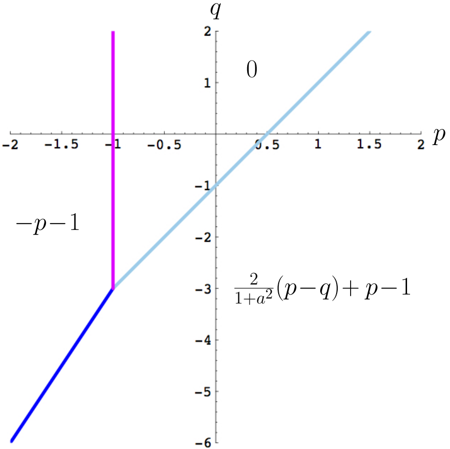

Figure: The three phases of the generalized integral means spectrum of the logarithmic spiral for $(p, q) \in \mathbb{R}^2$, with $\beta_{\text{tip}}(p; \kappa = 0) = -p - 1$, $\beta_0(p; \kappa = 0) = 0$, $\beta_1(p, q; \kappa = 0, a) = \frac{2}{1+a^2}(p - q) + p - 1$

Bounded SLE Integral Means Spectrum

- As predicted in D.'00 and Hastings '02, and proven in Beliaev & Smirnov '09, and Beliaev, D. & Zinsmeister '17, the **expected spectrum** of SLE_κ involves 3 phases:

$$\beta_{\mathrm{tip}}(p,\kappa) = -p - 1 + \frac{1}{4}\left(4+\kappa-\sqrt{(4+\kappa)^2-8\kappa p}\right),$$

$$\beta_0(p,\kappa) = -p + \frac{(4+\kappa)^2}{4\kappa} - \frac{(4+\kappa)}{4\kappa}\sqrt{(4+\kappa)^2-8\kappa p},$$

$$\beta_{\mathrm{lin}}(p,\kappa) = p - \frac{(4+\kappa)^2}{16\kappa}.$$

- a.s. β_{tip} [Johansson Viklund & Lawler '12]
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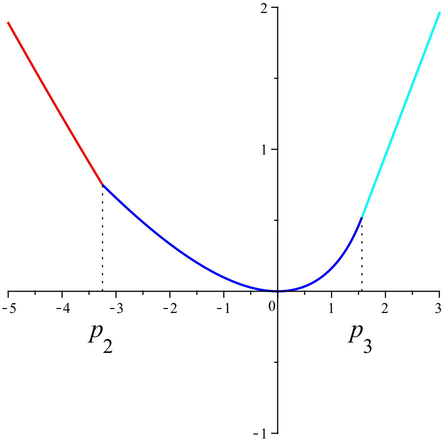
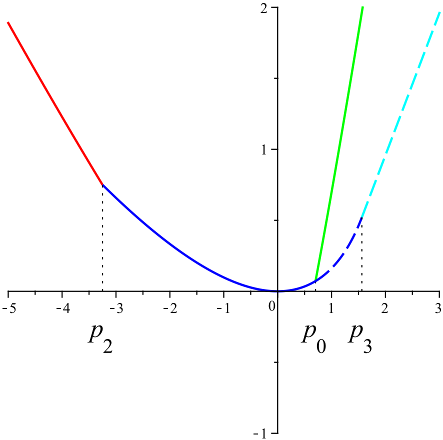


$$p_2 = -1 - \frac{3\kappa}{8}, \quad p_3 = \frac{3(4+\kappa)^2}{32\kappa}$$

Average integral means spectrum for **bounded** whole-plane SLE.



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Average integral means spectrum for **unbounded** whole-plane SLE.

Generalized Spectrum of Unbounded Whole-Plane SLE

- The generalized spectrum is [D., Ho, Le & Zinsmeister '18],

$$\beta_1(p, q; \kappa) := 3p - 2q - \frac{1}{2} - \frac{1}{2}\sqrt{1 + 2\kappa(p - q)}.$$

- Phase transition lines: **green parabola** & **blue quartic**

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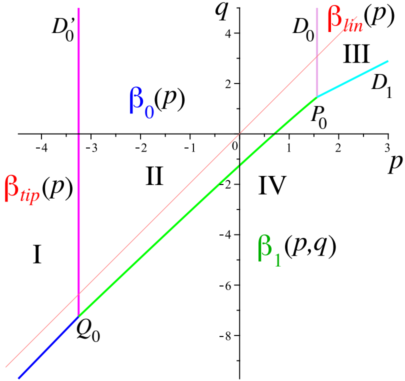

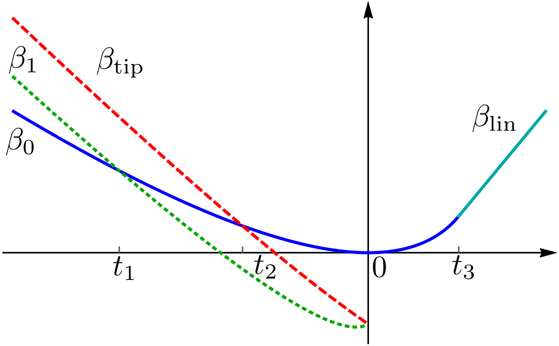

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'Second tip' spectrum

[Beliaev, D. & Zinsmeister '17]

Domain of Proof

- Domain where the form of the generalized integral means spectrum has been established [DHLZ '18], [Ho '22]

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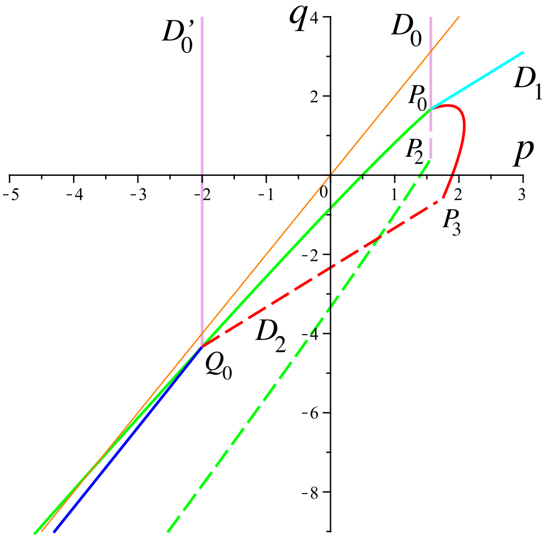

Complex Generalized Moments with Drift

- f **whole-plane (inner)** SLE_κ, $z \in \mathbb{D}$, $(p, q) \in \mathbb{C}^2$, $a \in \mathbb{R}$

$$F(z) := \mathbb{E}\left(f'(z)^{\frac{p}{2}}\left(\frac{z}{f(z)}\right)^{\frac{q}{2}}\right),\; G(z,\bar{z}) := \mathbb{E}\left|(f'(z))^{p}\left(\frac{z}{f(z)}\right)^{q}\right|.$$

- The differential equation for F is $\mathcal{P}(\partial)[F(z)] = 0$, with

$$\mathcal{P}(\partial) := -\frac{\kappa}{2}(z\partial_z)^2 - \left[\frac{1+z}{1-z} - ia\right] z\partial_z - \frac{p}{(1-z)^2} + \frac{q}{1-z} + p - q,$$

- while that for G is $\mathcal{P}(D)[G(z,\bar{z})] = 0$, with

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Complex Integrability

- Let f be a time 0 whole-plane (inner) SLE_κ, and $(p,q)\in\mathbb{C}^2$,

$$F(z) := \mathbb{E}\left(f'(z)^{\frac{p}{2}}\left(\frac{z}{f(z)}\right)^{\frac{q}{2}}\right),\ G(z,\bar z) := \mathbb{E}\left|f'(z)^{p}\left(\frac{z}{f(z)}\right)^{q}\right|.$$

- **Integrable complex parabola** with parameterization,

$$p(\gamma) := (2+\frac{\kappa}{2})\gamma - \frac{\kappa}{2}\gamma^2,\quad \gamma\in\mathbb{C},$$
$$q(\gamma) := (3 - ia + \frac{\kappa}{2})\gamma - \kappa\gamma^2.$$

- **Theorem** [DHNZ '23]: If $p = p(\gamma)$ and $q = q(\gamma)$, then

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- Let f be a time 0 whole-plane (inner) SLE_κ, and $(p, q) \in \mathbb{C}^2$,

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Complex Generalized Integral Means Spectrum with Drift

- The generalized **packing** spectrum for real p, q is

$$s_1(p,q;\kappa) := \beta_1(p,q;\kappa) - p + 1; \quad p,q \in \mathbb{R}$$
$$= 2(p-q) + \frac{1}{2} - \frac{1}{2}\sqrt{1+2\kappa(p-q)}$$

- **Complex moments** [D., Han, Nguyen & Zinsmeister '23]

$$s_1(p,q;\kappa) := \beta_1(p,q;\kappa) - \Re(p) + 1; \quad p,q \in \mathbb{C}$$
$$= 2\tau + \frac{1}{2} - \frac{1}{2}\sqrt{1+2\kappa\tau}$$

$$1 + 2\kappa\tau = \frac{1}{2}\left[1 + 2\kappa\Re(p-q) + |1+2\kappa(p-q)|\right]$$

$$(w.\,drift) = \frac{1}{2}\left[\Re[(1+ia)^2 + 2\kappa(p-q)] + |(1+ia)^2 + 2\kappa(p-q)|\right].$$

- Liouville quantum gravity & KPZ
D., Miller & Sheffield, *Astérisque* **427**, 2021

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Generalized Integral Means Spectrum, Drift $a, p, q \in \mathbb{R}$

$$\beta_1(p, q; \kappa, a) - p = 2\tau - \frac{1}{2} - \frac{1}{2}\sqrt{1 + 2\kappa\tau}, \; p - q = \tau\left(1 + \frac{a^2}{1 + 2\kappa\tau}\right)$$

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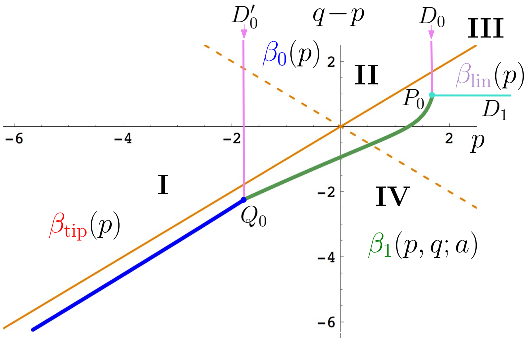

$\kappa = 2, a = 1$

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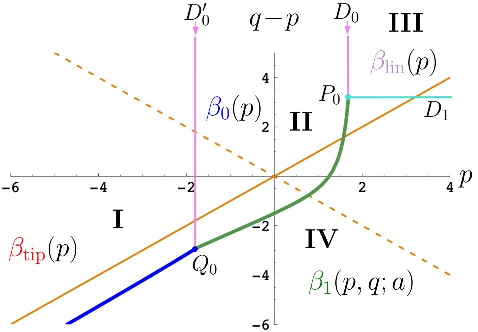

$\kappa = 2, a = 2$

Generalized Integral Means Spectrum, Drift $a, p, q \in \mathbb{R}$

$$\beta_1(p, q; \kappa, a) - p = 2\tau - \frac{1}{2} - \frac{1}{2}\sqrt{1 + 2\kappa\tau}, \; p - q = \tau\left(1 + \frac{a^2}{1 + 2\kappa\tau}\right)$$

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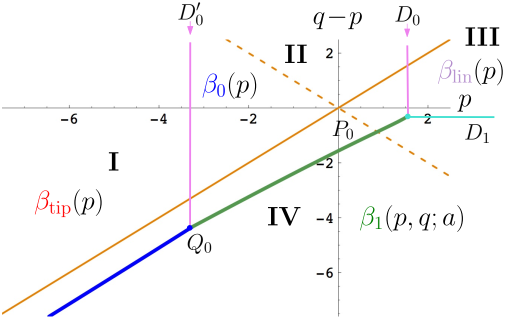

$\kappa = 6, a = 2$

Generalized Integral Means Spectrum, Drift $a, p, q \in \mathbb{R}$

$$\beta_1(p,q;\kappa,a) - p = 2\tau - \frac{1}{2} - \frac{1}{2}\sqrt{1+2\kappa\tau},\ p - q = \tau\left(1 + \frac{a^2}{1+2\kappa\tau}\right)$$

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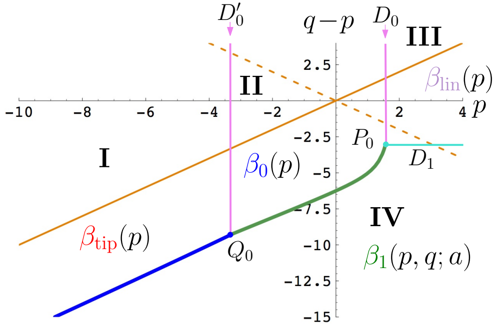

$\kappa = 6, a = 8$