Harmonizable Fractional Stable Motion: simultaneous estimators for the both parameters

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1. Introduction: comparison of HFSM with LFSM
2. The keystone of the talk
3. Main results and main lines of their proofs
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A real-valued harmonizable fractional stable motion (HFSM), denoted by $\{X(t)\}_{t\in\mathbb{R}}$, is a paradigmatic example of a continuous symmetric stable self-similar stochastic process with stationary increments: for any fixed $\tau \in \mathbb{R}$,

$$\{X(\tau+t) - X(\tau)\}_{t\in\mathbb{R}} \stackrel{d}{=} \{X(t) - X(0)\}_{t\in\mathbb{R}} = \{X(t)\}_{t\in\mathbb{R}}. \tag{1.1}$$

HFSM was introduced a long time ago in in the pioneering article (Cambanis and Maejima, 1989). A detailed presentation of it and many other related topics can be found in the well-known book on stable random variables and processes (Samorodnitsky and Taqqu,1994).

Basically, the HFSM $\{X(t)\}_{t\in\mathbb{R}}$ depends on two parameters: the Hurst parameter $H \in (0,1)$, and the stability parameter $\alpha \in (0,2]$.

Among other things, the parameter H governs roughness of sample paths of $\{X(t)\}_{t\in\mathbb{R}}$ and its self-similarity property:

$$\left\{a^{-H}X(at)\right\}_{t\in\mathbb{R}} \stackrel{d}{=} \{X(t)\}_{t\in\mathbb{R}}, \quad \text{for any fixed } a \in (0,+\infty). \tag{1.2}$$

Notice in passing that (1.2) implies that $X(0) \stackrel{a.s.}{=} 0$, this is why the last equality in (1.1) holds.

While the parameter α determines heaviness of tails of marginal distributions of $\{X(t)\}_{t\in\mathbb{R}}$: except in the very particular Gaussian case $\alpha = 2$ in which the probability $\mathbb{P}\big(|X(t)| \geq z\big)$ vanishes exponentially fast when $z \to +\infty$, for any other value of α and for each $t \neq 0$, one has, for some constants $0 < c'(t) < c''(t)$,

$$c'(t)z^{-\alpha} \leq \mathbb{P}\big(|X(t)| \geq z\big) \leq c''(t)z^{-\alpha}, \quad \text{for all } z \in [1, +\infty), \tag{1.3}$$

which implies that, for every $\beta \in (0, +\infty)$,

$$\mathbb{E}\big(|X(t)|^{\beta}\big) < +\infty, \quad \text{if and only if } \beta \in (0, \alpha). \tag{1.4}$$

The HFSM $\{X(t)\}_{t\in\mathbb{R}}$ is defined, for all $t \in \mathbb{R}$, through the stable stochastic integral in the frequency domain:

$$X(t) := \mathrm{Re}\left\{ \int_{\mathbb{R}} \frac{e^{it\xi} - 1}{|\xi|^{H+1/\alpha}} \, d\widetilde{M}_{\alpha}(\xi) \right\}, \tag{1.5}$$

where $\widetilde{M}_{\alpha}$ is a complex-valued rotationally invariant (i.e. $\forall\, \theta \in \mathbb{R}$, $e^{i\theta}\widetilde{M} \overset{d}{=} \widetilde{M}$) α-stable random measure with Lebesgue control measure.

The following remark will play a fundamental role in our talk.

Remark 1.1

(i) *The stable stochastic integral $\int_{\mathbb{R}} (\,\cdot\,)\, d\widetilde{M}_\alpha$ is a linear map on the Lebesgue space $L^\alpha(\mathbb{R})$ such that, for any deterministic function $g \in L^\alpha(\mathbb{R})$, the real part $\mathrm{Re}\left\{ \int_{\mathbb{R}} g(\xi)\, d\widetilde{M}_\alpha(\xi)\right\}$ is a real-valued Symmetric α-Stable (SαS) random variable with a scale parameter satisfying*

$$\sigma\left(\mathrm{Re}\left\{ \int_{\mathbb{R}} g(\xi)\, d\widetilde{M}_\alpha(\xi)\right\}\right)^\alpha = \int_{\mathbb{R}} \left|g(\xi)\right|^\alpha d\xi. \tag{1.6}$$

The equality (1.6) is reminiscent of the classical isometry property of Wiener integrals. We recall that a real-valued random variable X is SαS when its characteristic function $z \mapsto \mathbb{E}(e^{izX})$ satisfies $\mathbb{E}(e^{izX}) = e^{-\sigma^\alpha |z|^\alpha}$, for some $\sigma \in \mathbb{R}_+$ (called the scale parameter) and for all $z \in \mathbb{R}$.

(ii) *Let $m \in \mathbb{N}$ be arbitrary and let $f_1, \ldots, f_m$ be arbitrary functions of $L^\alpha(\mathbb{R})$ whose supports are disjoint up to Lebesgue negligible sets, then the real-valued SαS random variables $\mathrm{Re}\left\{ \int_{\mathbb{R}} f_1(\xi)\, d\widetilde{M}_\alpha(\xi)\right\}, \ldots, \mathrm{Re}\left\{ \int_{\mathbb{R}} f_m(\xi)\, d\widetilde{M}_\alpha(\xi)\right\}$ are independent.*

In the very particular Gaussian case $\alpha = 2$, the HFSM $\{X(t)\}_{t\in\mathbb{R}}$ reduces to the very classical Gaussian fractional Brownian motion (FBM)

$$B_H(t) = \mathrm{Re}\Big\{ \int_{\mathbb{R}} \frac{e^{it\xi}-1}{|\xi|^{H+1/2}}\, d\widetilde{M}_2(\xi) \Big\}. \tag{1.7}$$

The Gaussian process $\{B_H(t)\}_{t\in\mathbb{R}}$ can also be represented as a moving average stochastic Wiener integral in the time domain:

$$B_H(t) = c_H \int_{\mathbb{R}} \Big((t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \Big)\, dM_2(s). \tag{1.8}$$

However, when $\alpha \neq 2$, the HFSM $\{X(t)\}_{t\in\mathbb{R}}$ can no longer be represented as a moving average stable stochastic integral in the time domain. Actually, it is very different from the real-valued linear fractional stable motion (LFSM) $\{L(t)\}_{t\in\mathbb{R}}$ defined, for each $t\in\mathbb{R}$, by

$$L(t) := \int_{\mathbb{R}} \Big((t-s)_+^{H-1/\alpha} - (-s)_+^{H-1/\alpha} \Big)\, dM_\alpha(s), \tag{1.9}$$

where M_α is a real-valued α-stable random measure.

The large differences between the two processes $\{X(t)\}_{t\in\mathbb{R}}$ and $\{L(t)\}_{t\in\mathbb{R}}$ can be explained by several reasons. Two major ones of them are:

(1) Unlike the process $\{L(t)\}_{t\in\mathbb{R}}$ the process $\{X(t)\}_{t\in\mathbb{R}}$ fails to be ergodic.

(2) Behavior of sample paths of $\{L(t)\}_{t\in\mathbb{R}}$ and $\{X(t)\}_{t\in\mathbb{R}}$ is far from being the same.

Sample paths of $\{L(t)\}_{t\in\mathbb{R}}$ are multifractal functions (Balança, 2014) which become discontinuous when $H \le 1/\alpha$ and even unbounded on any interval when $H < 1/\alpha$ (see e.g. Samorodnitsky and Taqqu,1994).

While those of $\{X(t)\}_{t\in\mathbb{R}}$ are, for any value of $H \in (0,1)$, monofractal functions with pointwise and local Hölder exponents equal to H at any location (Ayache and Xiao, 2016). Also, for later purposes, we mention that as regards their behavior at infinity it has been shown in (Ayache and Boutard, 2017) that, for all fixed $\delta > 0$, one has, almost surely,

$$\sup_{|t|\ge 1}\left\{\frac{|X(t)|}{|t|^{H+\delta}}\right\} < +\infty\,. \tag{1.10}$$

Statistical estimators for the parameters H and α of the LFSM $\{L(t)\}_{t\in\mathbb{R}}$ and related moving average stable processes have already been proposed in several articles in the literature, some of which appeared a long time ago. The ergodicity property of the LFSM plays a crucial role in them. To illustrate the latter fact let us outline the method which allowed Taqqu and his co-authors to obtain a wavelet estimator for the parameter H of LFSM.

Let ψ be a "nice" real-valued function on $\mathbb{R}$. For all $(j,k)\in\mathbb{Z}_+^2$, one sets

$$\psi_{j,k}(t) := \psi(2^j t - k), \quad \text{for all } t\in\mathbb{R}. \tag{1.11}$$

The discrete wavelet transforms $\mathcal{W}_{j,k}(L)$, $(j,k)\in\mathbb{Z}_+^2$, of the LFSM are defined as

$$\mathcal{W}_{j,k}(L) := \int_{\mathbb{R}} \psi_{j,k}(t)L(t)\,dt = 2^{-j}\int_{\mathbb{R}} \psi(t'-k)L\big(2^{-j}t'\big)\,dt'. \tag{1.12}$$

For each fixed $\gamma\in(0,\alpha)$, the statistics $V_{j,\gamma}^{2^j}(L)$ is defined as

$$V_{j,\gamma}^{2^j}(L) := \sum_{k=1}^{2^j} \big|\mathcal{W}_{j,k}(L)\big|^{\gamma}. \tag{1.13}$$

The self-similarity property of the LFSM implies, for each fixed $j \in \mathbb{Z}_+$, that

$$V_{j,\gamma}^{2^j}(L) \stackrel{d}{=} 2^{-j(1+H)\gamma} V_{0,\gamma}^{2^j}(L) = 2^{-j(1+H)\gamma} \sum_{k=1}^{2^j} \left|\mathcal{W}_{0,k}(L)\right|^{\gamma}. \tag{1.14}$$

Moreover, the fundamental ergodicity property of the LFSM entails that the stationary process $\left\{\mathcal{W}_{0,k}^{(L)}\right\}_{k\in\mathbb{Z}_+}$ is ergodic. Thus, using the Birkhoff's ergodic Theorem, one gets that

$$2^{-j} V_{0,\gamma}^{2^j}(L) \xrightarrow[j\to+\infty]{a.s.} \mathbb{E}\Big(\left|\mathcal{W}_{0,0}(L)\right|^{\gamma}\Big). \tag{1.15}$$

Thus, one can derive from (1.14) that

$$\widetilde{H}_{j,\gamma}(L) := \gamma^{-1} - 1 - (\gamma j)^{-1} \log_2\big(V_{j,\gamma}^{2^j}(L)\big) \tag{1.16}$$

is a consistent estimator of the Hurst parameter H of the LFSM.

Unfortunately such a strategy is not applicable to the HFSM $\{X(t)\}_{t\in\mathbb{R}}$ since it is not ergodic. To the best of our knowledge no statistical estimator for any one of its two parameters H and α has been proposed in the literature so far.

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The keystone of our new strategy, for obtaining strongly consistent simultaneous estimators for the both parameters H and α of the HFSM $\{X(t)\}_{t\in\mathbb{R}}$, is to construct new transforms of it which allow to obtain, at any dyadic level $j \in \mathbb{N}$, a sequence $Y_{j,k}$, $k \in \mathbb{N}$, of *independent* real-valued SαS random variables whose scale parameters $\sigma(Y_{j,k})$, $k \in \mathbb{N}$, are connected in a rather simple way to the unknown parameters H and α.

These new transforms $Y_{j,k}$, $(j,k) \in \mathbb{N}^2$, of $\{X(t)\}_{t\in\mathbb{R}}$ are at the same time inspired by discrete wavelet transforms $\mathcal{W}_{j,k}$, $(j,k) \in \mathbb{Z}_+^2$, of $\{X(t)\}_{t\in\mathbb{R}}$ and significantly different from them. Indeed, while $\mathcal{W}_{j,k}$ is defined (sometimes up to normalizing factor), as

$$\mathcal{W}_{j,k} := \int_{\mathbb{R}} \psi_{j,k}(t) X(t)\, dt, \tag{2.1}$$

where $\psi_{j,k}(t) := \psi(2^j t - k)$, for all $t \in \mathbb{R}$. We define $Y_{j,k}$ as

$$Y_{j,k} := \frac{2}{2\pi} \int_{\mathbb{R}} \mathrm{Re}\left(\widehat{\psi}_{j,k}(t)\right) X(t)\, dt. \tag{2.2}$$

Our results are valid under general assumptions on the function $\psi : \mathbb{R} \to \mathbb{R}$:

($\mathcal{A}_1$) ψ is an even (i.e. $\forall\, \xi \in \mathbb{R}$, $\psi(-\xi) = \psi(\xi)$) continuous function on $\mathbb{R}$ with compact support included in the compact interval $I := [-2^{-1}, 2^{-1}]$.

($\mathcal{A}_2$) $\widehat{\psi}$ (the Fourier transform of ψ) is a real-valued, even, continuous function on $\mathbb{R}$ satisfying, for some constant c,

$$\big|\widehat{\psi}(t)\big| \le c\big(1 + |t|\big)^{-2}, \quad \text{for every } t \in \mathbb{R}. \tag{2.3}$$

There is no need to impose to ψ any vanishing moment condition. There are many functions satisfying ($\mathcal{A}_1$) and ($\mathcal{A}_2$), for instance the triangle function:

$$\psi(\xi) := 4\big(\mathbb{1}_{[-4^{-1},4^{-1}]} * \mathbb{1}_{[-4^{-1},4^{-1}]}\big)(\xi) = 2\big(1 - |2\xi|\big)\mathbb{1}_{[-2^{-1},2^{-1}]}(\xi), \quad \text{for all } \xi \in \mathbb{R}. \tag{2.4}$$

Notice that the equality $\psi_{j,k}(\xi) := \psi(2^j\xi - k)$, for all $\xi \in \mathbb{R}$ and ($\mathcal{A}_2$) entail that

$$\mathrm{Re}\,\big(\widehat{\psi}_{j,k}(t)\big) = 2^{-j}\cos(2^{-j}kt)\widehat{\psi}(2^{-j}t), \quad \text{for all } j,k \in \mathbb{N} \text{ and } t \in \mathbb{R}, \tag{2.5}$$

and also that the pathwise Lebesgue integral $Y_{j,k} := \dfrac{2}{2\pi}\displaystyle\int_{\mathbb{R}} \mathrm{Re}\,\big(\widehat{\psi}_{j,k}(t)\big)X(t)\,dt$ exists and is finite.

A nice stochastic integral representation for the r.v. $Y_{j,k}$, for any $(j,k) \in \mathbb{N}^2$:

Lemma 2.1

$$Y_{j,k} := \frac{2}{2\pi} \int_{\mathbb{R}} \mathrm{Re}\,\big(\widehat{\psi}_{j,k}(t)\big) X(t)\, dt \tag{2.6}$$

$$= \frac{1}{2\pi} \mathrm{Re}\left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \frac{(e^{it\xi}-1)\big(\widehat{\psi}_{j,k}(t)+\widehat{\psi}_{j,k}(-t)\big)}{|\xi|^{H+1/\alpha}}\, d\widetilde{M}_\alpha(\xi) \right) dt \right)$$

is a real-valued $S\alpha S$ random variable which can almost surely be expressed as

$$Y_{j,k} = \mathrm{Re}\left(\int_{I_{j,k}} \left(\frac{\psi_{j,k}(\xi)+\psi_{j,k}(-\xi)}{|\xi|^{H+1/\alpha}} \right) d\widetilde{M}_\alpha(\xi) \right), \tag{2.7}$$

where

$$I_{j,k} = \left[\frac{-k-2^{-1}}{2^j}, \frac{-k+2^{-1}}{2^j} \right] \bigcup \left[\frac{k-2^{-1}}{2^j}, \frac{k+2^{-1}}{2^j} \right]. \tag{2.8}$$

The $I_{j,k}$, $k \in \mathbb{N}$, being disjoint, the r.v. $Y_{j,k}$, $k \in \mathbb{N}$, are independent.

The very crucial Lemma 2.1 can of course be proved in a rigorous way. Let us here justify it by more or less heuristic arguments. Denote by $\mathcal{F}^{-1}$ the inverse Fourier transform from $L^2(\mathbb{R})$ to itself. It can be expressed, for any $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, as

$$\mathcal{F}^{-1}(g)(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{it\xi} g(t)\, dt, \quad \text{for all } \xi \in \mathbb{R}. \tag{2.9}$$

Now, imagine that one can use "a stochastic Fubini theorem" which allows to interchange the two integrals $\int_{\mathbb{R}} (\cdot)\, dt$ and $\int_{\mathbb{R}} (\cdot) d\widetilde{M}_\alpha(\xi)$ related with $Y_{j,k}$. Then

$$\begin{aligned} Y_{j,k} &= \mathrm{Re}\Big(\int_{\mathbb{R}} \Big(\frac{1}{2\pi} \int_{\mathbb{R}} \frac{(e^{it\xi}-1)\big(\widehat{\psi}_{j,k}(t) + \widehat{\psi}_{j,k}(-t)\big)}{|\xi|^{H+1/\alpha}}\, dt \Big)\, d\widetilde{M}_\alpha(\xi) \Big) \\ &= \mathrm{Re}\Big(\int_{\mathbb{R}} \Big(\frac{\mathcal{F}^{-1}(\widehat{\psi}_{j,k})(\xi) + \mathcal{F}^{-1}(\widehat{\psi}_{j,k})(-\xi) - 2\mathcal{F}^{-1}(\widehat{\psi}_{j,k})(0)}{|\xi|^{H+1/\alpha}} \Big)\, d\widetilde{M}_\alpha(\xi) \Big) \\ &= \mathrm{Re}\Big(\int_{\mathbb{R}} \Big(\frac{\psi_{j,k}(\xi) + \psi_{j,k}(-\xi) - 2\psi_{j,k}(0)}{|\xi|^{H+1/\alpha}} \Big)\, d\widetilde{M}_\alpha(\xi) \Big). \end{aligned} \tag{2.10}$$

Moreover, $\psi_{j,k}(0) = 0$, since $0 \notin \operatorname{supp} \psi_{j,k} = \big[2^{-j}(k-2^{-1}), 2^{-j}(k+2^{-1})\big]$.

The scale parameters $\sigma(Y_{j,k})$'s of the SαS r.v. $Y_{j,k}$'s are connected in a rather simple way to the unknown parameters H and α of the HFSM. Indeed:

Lemma 2.2

One has

$$\sigma(Y_{j,k}) \asymp 2^{jH} k^{-(H+1/\alpha)}. \tag{2.11}$$

That is, there are two constants $0 < c_1 < c_2$ *such that, for all* $(j,k) \in \mathbb{N}^2$,

$$c_1\, 2^{jH} k^{-(H+1/\alpha)} \le \sigma(Y_{j,k}) \le c_2\, 2^{jH} k^{-(H+1/\alpha)}. \tag{2.12}$$

Proof Lemma 2.1, isometry property of stable stochastic integral and the fact that ψ is an even function with support in $[-2^{-1}, 2^{-1}]$ imply that

$$\begin{aligned}
\sigma(Y_{j,k})^\alpha &= \sigma\left(\mathrm{Re}\left(\int_{I_{j,k}} \left(\frac{\psi_{j,k}(\xi) + \psi_{j,k}(-\xi)}{|\xi|^{H+1/\alpha}}\right) d\widetilde{M}_\alpha(\xi)\right)\right)^\alpha \\
&= \int_{I_{j,k}} \frac{\left|\psi(2^j\xi - k) + \psi(-2^j\xi - k)\right|^\alpha}{|\xi|^{\alpha H+1}}\, d\xi \\
&= 2\int_{2^{-j}(k-2^{-1})}^{2^{-j}(k+2^{-1})} \frac{\left|\psi(2^j\xi - k)\right|^\alpha}{(\xi)^{\alpha H+1}}\, d\xi. \qquad (2.13)
\end{aligned}$$

Then, using the change of variable $\eta = 2^j\xi - k$ i.e. $\xi = 2^{-j}(\eta + k)$, one gets that

$$\sigma(Y_{j,k})^\alpha = 2^{j\alpha H+1} \int_{-2^{-1}}^{2^{-1}} \frac{\left|\psi(\eta)\right|^\alpha}{(\eta + k)^{\alpha H+1}}\, d\eta, \tag{2.14}$$

which clearly implies that

$$2^{j\alpha H+1}\left(k + 2^{-1}\right)^{-(\alpha H+1)} \int_{-2^{-1}}^{2^{-1}} \left|\psi(\eta)\right|^\alpha d\eta \le \sigma(Y_{j,k})^\alpha \tag{2.15}$$

and

$$\sigma(Y_{j,k})^\alpha \le 2^{j\alpha H+1}\left(k - 2^{-1}\right)^{-(\alpha H+1)} \int_{-2^{-1}}^{2^{-1}} \left|\psi(\eta)\right|^\alpha d\eta. \tag{2.16}$$

Finally the lemma results from (2.14), (2.15) and (2.16) and the two inequalities:

$$2^{-(\alpha H+1)} k^{-(\alpha H+1)} \le \left(k + 2^{-1}\right)^{-(\alpha H+1)}, \quad \text{for all } k \in \mathbb{N}, \tag{2.17}$$

and

$$\left(k - 2^{-1}\right)^{-(\alpha H+1)} \le 2^{(\alpha H+1)} k^{-(\alpha H+1)}. \tag{2.18}$$

□

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Throughout this section, one denotes by $(m_j)_{j\in\mathbb{N}}$ an arbitrary non-decreasing sequence (that is $m_j \le m_{j+1}$, for all $j \in \mathbb{N}$) of integers larger than 2 which always satisfy the condition

$$m_j \ge j\,, \quad \text{for all } j \in \mathbb{N}. \tag{3.1}$$

Definition 3.1

For each fixed $\gamma > 0$ small enough, and for every $j \in \mathbb{N}$, the statistics $V_{j,\gamma}^{m_j}$ is defined as

$$V_{j,\gamma}^{m_j} := \sum_{k=1}^{m_j} \left| Y_{j,k} \right|^{\gamma}. \tag{3.2}$$

Denoting by $\log_2$ the binary logarithm (i.e. $\log_2(2) = 1$), for all $j \in \mathbb{N}$, the statistics $V_{j,\log_2}^{m_j}$ is defined as

$$V_{j,\log_2}^{m_j} := \sum_{k=1}^{m_j} \log_2 \left| Y_{j,k} \right|. \tag{3.3}$$

Theorem 3.1

One assumes that $\alpha \in [\underline{\alpha}, 2]$, *where the lower bound* $\underline{\alpha} \in (0,2]$ *is known, and that* $\gamma \in (0, 4^{-1}\underline{\alpha})$. *For each* $j \in \mathbb{N}$, *one sets*

$$\widehat{H}_{j,\gamma} := \frac{\log_2(V_{j,\gamma}^{m_j})}{\gamma j} \quad \text{and} \quad \widehat{\alpha}_{j,\gamma} := \frac{\gamma j}{j - \log_2(V_{j,\gamma}^{m_j})}. \tag{3.4}$$

Then, the following two results hold.

(i) *Under the condition*

$$\lim_{j \to +\infty} \frac{\log_2(m_j)}{j} = 0, \tag{3.5}$$

$\widehat{H}_{j,\gamma}$ *is a strongly consistent (almost surely convergent) estimator of the Hurst parameter* H *of HFSM.*

(ii) *Under the condition*

$$\lim_{j \to +\infty} \frac{\log_2(m_j)}{j} = 1, \tag{3.6}$$

$\widehat{\alpha}_{j,\gamma}$ *is a strongly consistent estimator of the stability parameter* α *of HFSM.*

Theorem 3.2

For each $j \in \mathbb{N}$, one sets

$$\widehat{H}_{j,\log_2} := \frac{V_{j,\log_2}^{m_j}}{j\, m_j} \quad \text{and} \quad \widehat{\alpha}_{j,\log_2} := \frac{j\, m_j}{V_{j,\log_2}^{m_j}}. \tag{3.7}$$

Then, the following two results hold.

(i) *Under the same condition (3.5) as in Theorem 3.1, $\widehat{H}_{j,\log_2}$ is a strongly consistent estimator of the Hurst parameter H of HFSM.*

(ii) *Under the same condition (3.6) as in Theorem 3.1, $\widehat{\alpha}_{j,\log_2}$ is a strongly consistent estimator of the stability parameter α of HFSM.*

We will only sketch the proof of Theorem 3.1 since that of Theorem 3.2 can be done by following rather similar ideas.

The two main steps of the proof of Theorem 3.1:

• **Step 1:** One shows that the theorem is valid when $V_{j,\gamma}^{m_j}$ is replaced by $\mathbb{E}(V_{j,\gamma}^{m_j})$.
Recall that, for each fixed $\gamma \in (0,\alpha)$, there exists a universal positive finite constant $c_0(\gamma,\alpha)$ such that, for any SαS r.v. Z, one has

$$\mathbb{E}\big(|Z|^{\gamma}\big) = c_0(\gamma,\alpha)\sigma(Z)^{\gamma}. \tag{3.8}$$

Therefore

$$\mathbb{E}(V_{j,\gamma}^{m_j}) = c_0(\gamma,\alpha)\sum_{k=1}^{m_j}\sigma(Y_{j,k})^{\gamma}. \tag{3.9}$$

From now on $\gamma \in (0, 4^{-1}\underline{\alpha})$ which implies that $0 < \gamma(H+1/\alpha) < 3/4$. Then, in view of the fact that $\sigma(Y_{j,k})^{\gamma} \asymp 2^{j\gamma H} k^{-\gamma(H+1/\alpha)}$, one can derive from (3.9) that

$$\mathbb{E}(V_{j,\gamma}^{m_j}) \asymp 2^{j\gamma H} m_j^{1-\gamma(H+1/\alpha)} = 2^{j\left(\gamma H + j^{-1}\log_2(m_j)(1-\gamma(H+1/\alpha))\right)}. \tag{3.10}$$

Thus, when $\lim_{j\to+\infty} j^{-1}\log_2(m_j) = 0$, one has $\mathbb{E}(V_{j,\gamma}^{m_j}) \asymp 2^{j\gamma H}$, and when $\lim_{j\to+\infty} j^{-1}\log_2(m_j) = 1$, one has $\mathbb{E}(V_{j,\gamma}^{m_j}) \asymp 2^{j(1-\gamma/\alpha)}$. Then, the desired result follows from easy calculations.

- **Step 2:** By using Borel-Cantelli Lemma, one shows that

$$\frac{V_{j,\gamma}^{m_j}}{\mathbb{E}(V_{j,\gamma}^{m_j})} \xrightarrow[j\to+\infty]{a.s.} 1. \tag{3.11}$$

It follows from easy calculations and Markov inequality that, for each fixed $\varepsilon > 0$,

$$\mathbb{P}\Big(\Big|\frac{V_{j,\gamma}^{m_j}}{\mathbb{E}(V_{j,\gamma}^{m_j})} - 1\Big| \geq \varepsilon\Big) \leq \frac{\mathbb{E}\Big(\big|V_{j,\gamma}^{m_j} - \mathbb{E}(V_{j,\gamma}^{m_j})\big|^4\Big)}{\big(\varepsilon\,\mathbb{E}(V_{j,\gamma}^{m_j})\big)^4}. \tag{3.12}$$

One knows from the Step 1 that $\mathbb{E}(V_{j,\gamma}^{m_j}) \asymp 2^{j\gamma H} m_j^{1-\gamma(H+1/\alpha)}$, and consequently that, one has, for some constant $c_1 > 0$,

$$\big(\mathbb{E}(V_{j,\gamma}^{m_j})\big)^4 \geq c_1\, 2^{4j\gamma H} m_j^{4(1-\gamma(H+1/\alpha))}. \tag{3.13}$$

The main difficulty is to derive an appropriate upper bound for the expectation $\mathbb{E}\Big(\big|V_{j,\gamma}^{m_j} - \mathbb{E}(V_{j,\gamma}^{m_j})\big|^4\Big)$. It is at this stage that the independence property of the r.v. $Y_{j,k}$, $1 \leq k \leq m_j$, plays a very crucial role.

Indeed, it implies that the centered r.v. $\widetilde{Y}_{j,k,\gamma} := |Y_{j,k}|^\gamma - \mathbb{E}(|Y_{j,k}|^\gamma)$, $k \in \mathbb{N},, 1 \leq k \leq m_j$, are independent as well. Then, using the fact that

$$V_{j,\gamma}^{m_j} - \mathbb{E}(V_{j,\gamma}^{m_j}) = \sum_{k=1}^{m_j} \widetilde{Y}_{j,k,\gamma}, \tag{3.14}$$

one obtains that

$$\begin{aligned} \mathbb{E}\Big(\big|V_{j,\gamma}^{m_j} - \mathbb{E}(V_{j,\gamma}^{m_j})\big|^4\Big) &= \sum_{k_1,\ldots,k_4=1}^{m_j} \mathbb{E}\Big(\prod_{l=1}^{4} \widetilde{Y}_{j,k_l,\gamma}\Big) \\ &= \sum_{k=1}^{m_j} \mathbb{E}\big(\widetilde{Y}_{j,k,\gamma}^4\big) + \binom{4}{2} \sum_{1\leq k_1\neq k_2\leq m_j} \mathbb{E}\big(\widetilde{Y}_{j,k_1,\gamma}^2\big)\mathbb{E}\big(\widetilde{Y}_{j,k_2,\gamma}^2\big) \\ &\leq \sum_{k=1}^{m_j} \mathbb{E}\big(\widetilde{Y}_{j,k,\gamma}^4\big) + 6\Big(\sum_{k=1}^{m_j} \mathbb{E}\big(\widetilde{Y}_{j,k,\gamma}^2\big)\Big)^2, \end{aligned} \tag{3.15}$$

where the last equality follows from the fact that the r.v. $\widetilde{Y}_{j,k,\gamma}$, $1 \leq k \leq m_j$, are independent and centered.

There are two constants $c_2 > 0$ and $c_3 > 0$ such that, for all $(j,k) \in \mathbb{N}^2$,

$$\mathbb{E}\big(\widetilde{Y}_{j,k,\gamma}^4\big) = c_2 \sigma(Y_{j,k})^{4\gamma} \quad \text{and} \quad \mathbb{E}\big(\widetilde{Y}_{j,k,\gamma}^4\big) = c_3 \sigma(Y_{j,k})^{2\gamma}. \tag{3.16}$$

One can derive from (3.15), (3.16) and $\sigma(Y_{j,k})^{2\gamma} \asymp 2^{2j\gamma H} k^{-2\gamma(H+1/\alpha)}$ that

$$\begin{aligned} \mathbb{E}\Big(\big|V_{j,\gamma}^{m_j} - \mathbb{E}(V_{j,\gamma}^{m_j})\big|^4\Big) &\leq c_2 \sum_{k=1}^{m_j} \sigma(Y_{j,k})^{4\gamma} + 6c_3^2 \Big(\sum_{k=1}^{m_j} \sigma(Y_{j,k})^{2\gamma}\Big)^2 \\ &\leq c_4 \Big(\sum_{k=1}^{m_j} \sigma(Y_{j,k})^{2\gamma}\Big)^2 \leq c_5 2^{4j\gamma H} \Big(\sum_{k=1}^{m_j} k^{-2\gamma(H+1/\alpha)}\Big)^2 \\ &\leq c_6 2^{4j\gamma H} \Big(1 + m_j^{2(1-2\gamma(H+1/\alpha))} + \log^2(m_j)\Big). \end{aligned} \tag{3.17}$$

Finally, combining (3.17) with $\big(\mathbb{E}(V_{j,\gamma}^{m_j})\big)^4 \geq c_1\, 2^{4j\gamma H} m_j^{4(1-\gamma(H+1/\alpha))}$ and $m_j \geq j$, one gets

$$\sum_{j=1}^{+\infty} \mathbb{P}\Big(\Big|\frac{V_{j,\gamma}^{m_j}}{\mathbb{E}(V_{j,\gamma}^{m_j})} - 1\Big| \geq \varepsilon\Big) \leq \sum_{j=1}^{+\infty} \frac{\mathbb{E}\Big(\big|V_{j,\gamma}^{m_j} - \mathbb{E}(V_{j,\gamma}^{m_j})\big|^4\Big)}{\big(\varepsilon\, \mathbb{E}(V_{j,\gamma}^{m_j})\big)^4} < +\infty. \tag{3.18}$$

Organization of the talk

1. Introduction: comparison of HFSM with LFSM
2. The keystone of the talk
3. Main results and main lines of their proofs
4. Three perspectives

1) We believe that the new strategy introduced in our talk would open the door to statistical estimation of parameters of harmonizable stable fields extending the HFSM, as for instance the isotropic harmonizable fractional stable field or the anisotropic harmonizable fractional stable sheet.

2) Since, for each for each fixed $j \in \mathbb{N}$, the random variables $Y_{j,k}$, $k \in \mathbb{N}$, have the very nice property to be independent, we believe that the centered and standardized versions of the two statistics $V_{j,\gamma}^{m_j}$ and $V_{j,\log_2}^{m_j}$ would be asymptotically normal. This would imply that the four estimators $\widehat{H}_{j,\gamma}$, $\widehat{\alpha}_{j,\gamma}$, $\widehat{H}_{j,\log_2}$ and $\widehat{\alpha}_{j,\log_2}$ would be asymptotically normal as well.

3) In our talk, the four estimators $\widehat{H}_{j,\gamma}$, $\widehat{\alpha}_{j,\gamma}$, $\widehat{H}_{j,\log_2}$ and $\widehat{\alpha}_{j,\log_2}$ are obtained from the observation of a sample path of the HFSM $\{X(t)\}_{t\in\mathbb{R}}$ in continuous time, we believe that it would be possible to extend our estimation procedures to a framework where only a discretized sample path of $\{X(t)\}_{t\in\mathbb{R}}$ is observed.