Quasi-stationary distributions without killing

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Joint work with R. Fernandez, F. Manzo and E. Scoppola

Analysis and simulations of metastable systems

CIRM, Luminy

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Sapienza Università di Roma

Setting

Consider

- A discrete time Markov chain $(X_t)_{t\geq 0}$
- on a state space Ω of size n,
- with transition matrix P,
- P reversible w.r.t. the equilibrium distribution π,

$$\pi = \pi P \qquad \text{and} \qquad \pi(x)P(x,y) = \pi(y)P(y,x)$$

- P has positive eigenvalues,
- fix an initial distribution α.

Question

What is the distribution of the chain in the long run, $t \gg 1$, conditioning on the event

"X_t is not at equilibrium" ?

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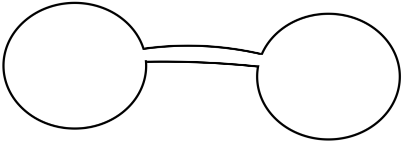
Two cartoons

Entropy driven metastability



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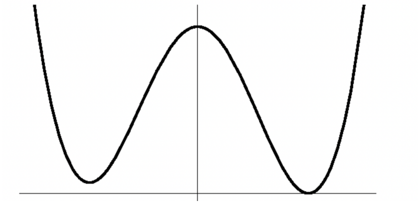

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When t is "large"

$$\{\text{not at equilibrium at time } t\} \approx \{\tau_G > t\}$$

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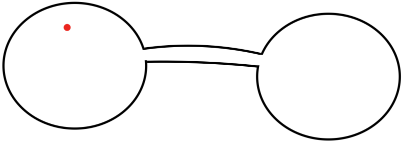

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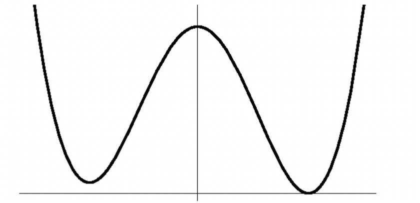

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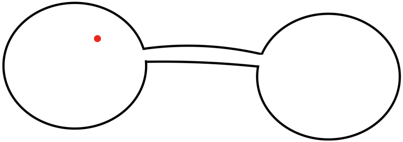

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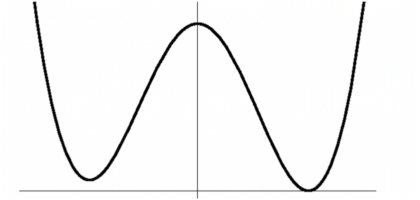

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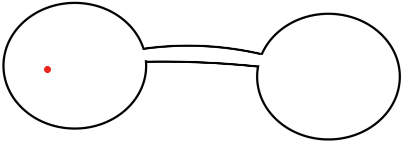

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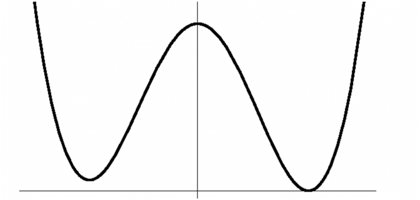

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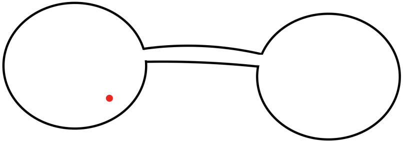

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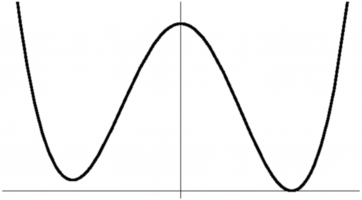

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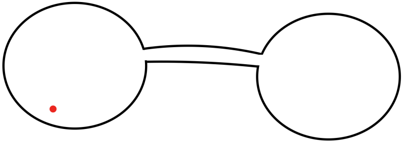

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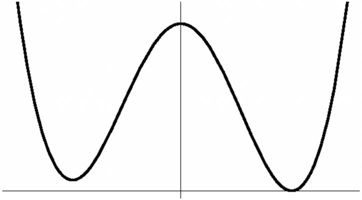

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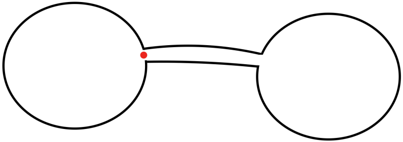

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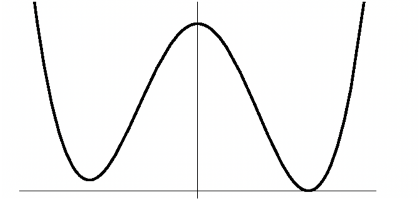

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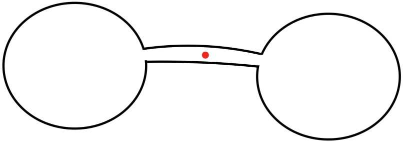

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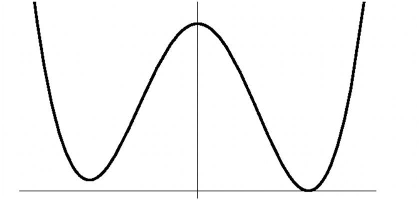

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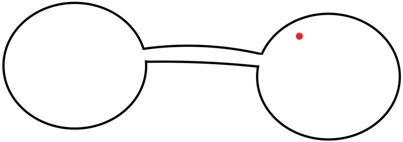

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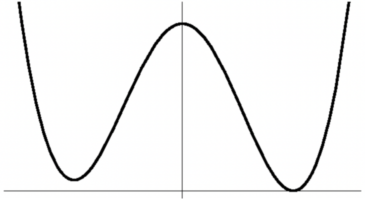

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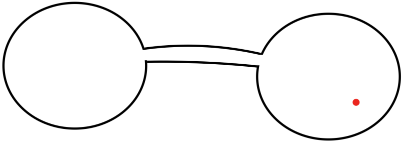

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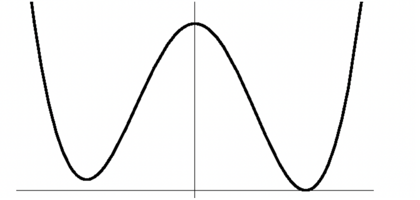

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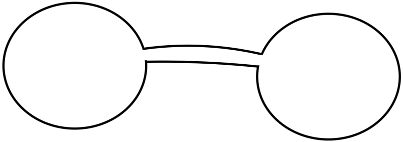

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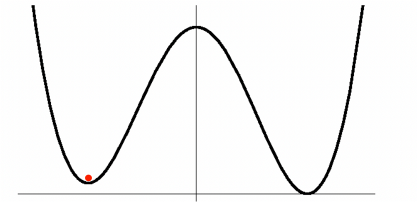

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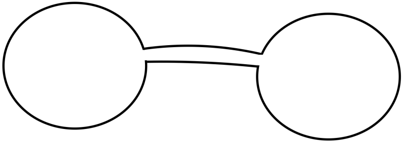

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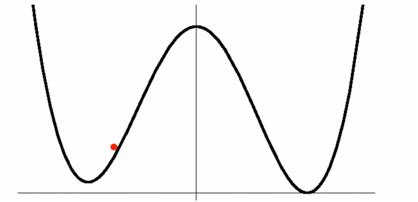

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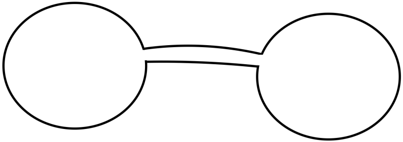

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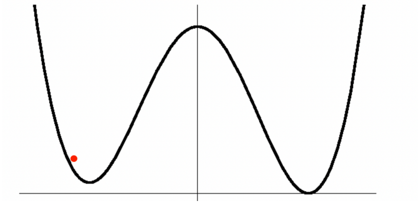

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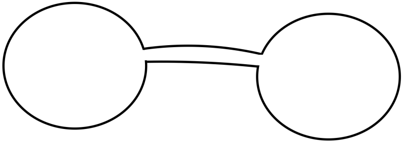

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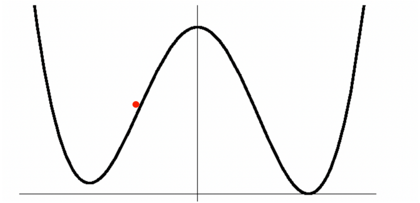

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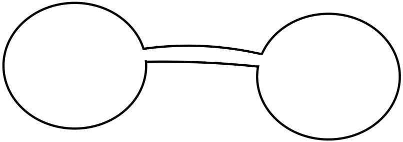

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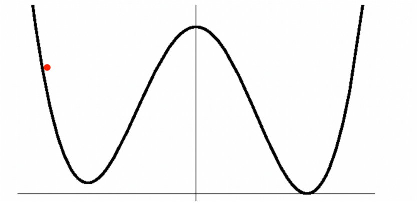

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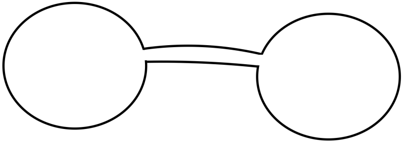

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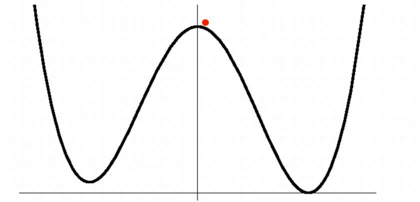

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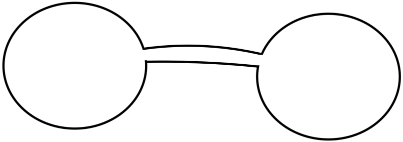

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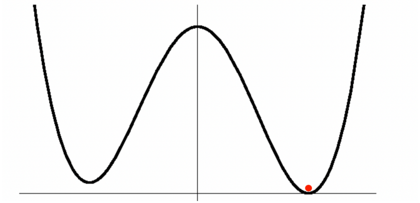

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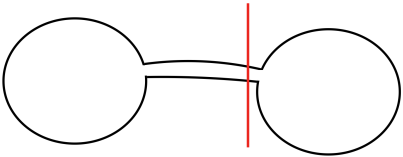

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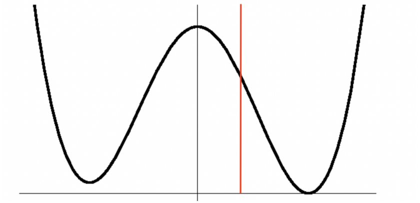

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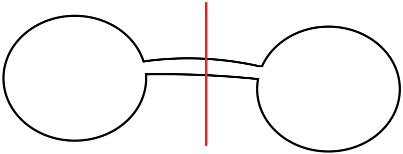

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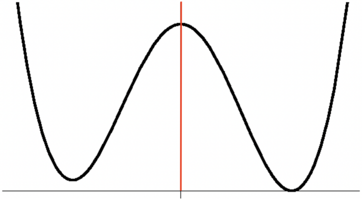

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Fix a target set: $G \subset \Omega$.

- $[P]_G$ the restricted matrix in which rows/columns relative to G are erased.
- $[P]_G$ is a sub-stochastic matrix: The process is **killed** in G.
- Fix α a probability distribution on $\Omega \setminus G$.
- There exists a probability distribution $\mu_\star^\alpha$ on $\Omega \setminus G$ such that

$$\lim_{t\to\infty} \mathbb{P}_\alpha(X_t = x \mid \tau_G > t) = \mu_\star^\alpha(x)\,, \qquad \forall x \in \Omega \setminus G.$$

- $\mu_\star$ is the quasi-stationary distribution associated to α (and to the target G).
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Optimal Strong Stationary Times (OSST)

- For a given initial distribution α let

$$\mu_t^\alpha(x) := \sum_{y\in\Omega} \alpha(y) P^t(y,x), \qquad \forall x \in \Omega.$$

- We will use the separation distance to quantify the convergence to stationarity

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and, further,

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QSD without killing

Consider the sequence of conditional distributions $(\varphi_t^\alpha)_{t\geq 0}$, defined as follows:

$$\varphi_t^\alpha(y) := \mathbb{P}_\alpha\left(X_t = y \mid \tau_\pi^\alpha > t\right), \qquad \forall t \geq 0,\ y \in \Omega. \tag{1}$$

Theorem

If the transition matrix P is reversible, irreducible and with positive eigenvalues, then for every starting distribution $\alpha \neq \pi$ there exists the limit

$$\varphi_\star^\alpha(y) := \lim_{t\to\infty} \varphi_t^\alpha(y), \qquad \forall y \in \Omega.$$

Characterization of $\varphi_\star^\alpha$

For every α there exists a couple $(v^\alpha, \lambda^\alpha) \in \mathbb{R}^{|\Omega|} \times (0,1)$ such that

1. $(v^\alpha, \lambda^\alpha)$ satisfy
$$v^\alpha P = \lambda^\alpha v^\alpha.$$
2. The distribution $\varphi_\star^\alpha$ is in the span of v^α and π. In particular
$$\varphi_\star^\alpha = v^\alpha + \pi.$$
3. The distribution $\varphi_\star^\alpha$ satisfies
$$\varphi_\star^\alpha P = \lambda^\alpha \varphi_\star^\alpha + \left(1 - \lambda^\alpha\right)\pi.$$
4. The separation distance, starting at α, decays exponentially at rate λ^α, i.e.,
$$\lim_{t\to\infty} \left(s_t^\alpha\right)^{\frac{1}{t}} = \lambda^\alpha.$$
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An explicit formula for $\varphi_\star^\alpha$

By spectral decomposition

$$P = |1\rangle\langle\pi| + \sum_{i=2}^{n} \lambda_i |f_i\rangle\langle g_i|, \qquad \langle\alpha| = \langle\pi| + \sum_{i=2}^{n} c_i \langle g_i|.$$

Define

$$\lambda^\alpha := \max_{i\in\{2,\dots,n\}} \{\lambda_i \text{ s.t. } c_i \neq 0\} \quad \text{and} \quad \mathcal{I}^\alpha := \{i \in \{2,\dots,n\} \text{ s.t. } \lambda_i = \lambda^\alpha\}.$$

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$$\langle\varphi_\star^\alpha| = \langle v^\alpha| + \langle\pi| = \frac{1}{\ell^\alpha}\sum_{i\in\mathcal{I}^\alpha} c_i \langle g_i| + \langle\pi|$$

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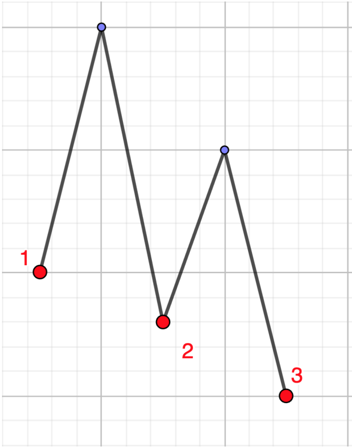
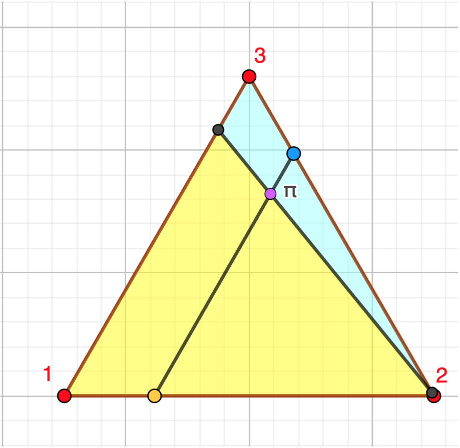
A toy example





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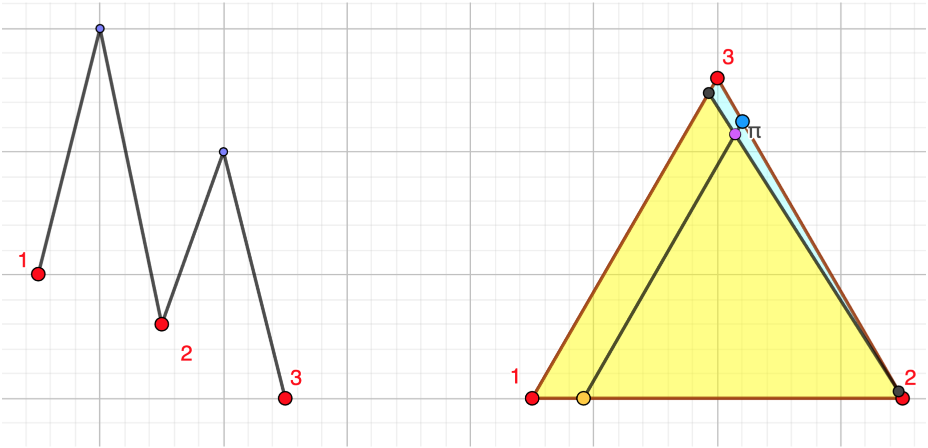

Conclusions

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2. Such QSDs share the same set of features of the classical QSD, i.e.,
    - Can be read as a Yaglom limit.
    - Can be characterized in a linear-algebraic way.
    - Exhibit an exponential law of the "hitting of the target", i.e., π.
    - The rate of the exponential is an eigenvalue of P.
3. Open problem:

   Is there a natural notion of Doob's transform associated to $\varphi_\star^\alpha$?

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Thank you!