Metastability for Glauber dynamics with inhomogeneous coupling disorder

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(with A. Bovier, F. den Hollander, S. Marello and M. Slowik)

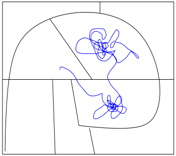
Analysis and Simulations of Metastable Systems
CIRM, Marseille
3–7 April 2023

TU Delft Delft University of Technology

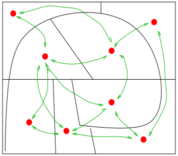
What is metastability?

Metastability is a phenomenon where a system, under the influence of a stochastic dynamics, moves between different regions of its state space on different time scales.

Fast time scale: quasi-equilibrium within single subregion



Slow time scale: transitions between different subregions



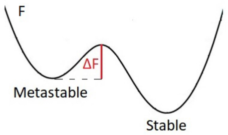
In Physics: metastability is the dynamical manifestation of a phase transition.

How to study metastability?

Given F the free energy, the quantities of interest are

1. Metastable parameter regime $\rightarrow$ multiple minima of F
2. Critical points of F:
   - local minima (metastable states)
   - global minimum (stable state)
   - local maxima/saddle points
3. **Mean hitting time**: the mean time the system (subject to a stochastic dynamics) needs to "hit" the stable state starting from a metastable state.

   Arrhenius law: $\mathbb{E}(\tau) \sim \exp(N\Delta F)$, in the limit $N \rightarrow \infty$.



Monographs:

- Olivieri and Vares 2005
- Bovier and den Hollander 2015

The quenched model

Mean-field version of Ising type spin model with inhomogeneous bond disorder.
N spins, $[N] = \{1, 2, \ldots, N\}$
Configuration space $\mathcal{S}_N = \{-1, +1\}^N$
Configuration $\sigma = (\sigma_i)_{i \in [N]} \in \mathcal{S}_N$, $\sigma_i \in \{-1, +1\}$
$h > 0$ constant magnetic field.
Hamiltonian of quenched/dilute model

$$H_N^{\mathsf{que}}(\sigma) = -\frac{1}{N} \sum_{1 \le i < j \le N} J_{ij} \sigma_i \sigma_j - h \sum_{i=1}^{N} \sigma_i$$

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Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $(J_{ij})_{1 \le i,j < \infty}$ a triangular array of random variables.

The quenched model

Hamiltonian

$$H_N^{\text{que}}(\sigma) = -\frac{1}{N} \sum_{1 \le i < j \le N} J_{ij} \sigma_i \sigma_j - h \sum_{i=1}^{N} \sigma_i$$

We take $\boxed{J_{ij} = A_{ij} B_{ij}}$

where $(A_{ij}), (B_{ij}), (P_{ij})$ are sequences of random variables such that

- $|A_{ij}| \le a, \quad B_{ij} \in [0, b], \quad P_{ij} \in (0, 1], \quad \mathbb{P}$-a.s., $\forall\, i, j$
- (B_{ij}) are mutually independent
- $B_{ij} \perp\!\!\!\perp \mathcal{F}_A$ and $\mathbb{E}\big[B_{ij} \mid \mathcal{F}_P\big] = P_{ij}$ $\quad \mathbb{P}$-a.s., $\forall\, i, j$

Notation: $\mathcal{F}_A$ and $\mathcal{F}_P$ are the σ-algebras generated by (A_{ij}) and (P_{ij}).

The quenched model

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Three possible randomness levels:

1. B_{ij} random
2. $\mathbb{E}\big[B_{ij} \mid \mathcal{F}_P\big] = P_{ij}$ random (quenched on this)
3. A_{ij} random (quenched on this)

Example: Ising model on random graphs

$J_{ij} = A_{ij} B_{ij}$,

$$H_N^{\mathsf{que}}(\sigma) = -\frac{1}{N} \sum_{1 \le i < j \le N} J_{ij} \sigma_i \sigma_j - h \sum_{1 \le i \le N} \sigma_i,$$

Example: Ising model on random graphs

$$J_{ij} = A_{ij}B_{ij}, \quad \text{if } A_{ij} \equiv 1,\ B_{ij} \sim \mathsf{Be}(P_{ij}) \in \{0,1\}$$

$$H_N^{\mathsf{que}}(\sigma) = -\frac{1}{N} \sum_{1\leq i<j\leq N} B_{ij}\sigma_i\sigma_j - h \sum_{1\leq i\leq N} \sigma_i,$$

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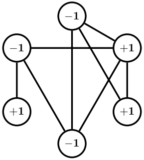
$$= -\frac{1}{N} \sum_{(i,j) \in \mathcal{E}} \sigma_i \sigma_j - h \sum_{1 \le i \le N} \sigma_i$$



The **interaction graph** $G = ([N], \mathcal{E})$: $(i,j) \in \mathcal{E} \iff B_{ij} \neq 0.$

$\implies$ Ising model on dense (inhomogeneous) random graphs: $\mathbb{P}((i,j) \in \mathcal{E}) = \mathbb{P}(B_{ij} = 1) = P_{ij}$.

Examples:

- Curie–Weiss model: $B_{ij} \equiv 1 \implies$ Ising on the complete graph
- $B_{ij} \sim \mathrm{Be}(p)$ i.i.d. $\implies$ $(P_{ij} \equiv p)$ Ising on the Erdős–Rényi random graph
- $B_{ij} \sim \mathrm{Be}(V_i V_j)$, V_i i.i.d. $\implies$ $(P_{ij} \equiv V_i V_j)$ Ising on the Chung-Lu r. g.

Example: dilute Hopfield model

Hopfield model, with n patterns.

Let $(\xi_i)_{i\in\mathbb{N}}$, $\xi_i \in \{-1,1\}^n$ *random* (typically $\mathbb{P}(\xi_i^k = 1) = \frac{1}{2}$):

$$\xi = \begin{pmatrix} +1 & -1 & -1 & +1 & -1 & -1 & -1 & \dots \\ -1 & -1 & -1 & +1 & -1 & +1 & -1 & \dots \\ -1 & +1 & -1 & -1 & +1 & -1 & +1 & \dots \\ -1 & -1 & -1 & +1 & +1 & -1 & -1 & \dots \\ +1 & +1 & +1 & +1 & -1 & +1 & -1 & \dots \end{pmatrix}$$

$$H_N^{\text{Hop}}(\sigma) = -\frac{1}{N} \sum_{1\le i<j\le N} \sum_{k=1}^{n} \sigma_i \sigma_j \xi_i^k \xi_j^k \qquad \sigma \in \{-1,1\}^N$$

[Anton Bovier, Veronique Gayrard, Pierre Picco...]

Example: dilute Hopfield model

Randomly dilute Hopfield model, with n patterns.

Let $(\xi_i)_{i\in\mathbb{N}}$, $\xi_i \in \{-1,1\}^n$ *random* (typically $\mathbb{P}(\xi_i^k = 1) = \frac{1}{2}$):

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$J_{ij} = A_{ij}B_{ij}$. Taking $A_{ij} = \sum_{1\le k\le n} \xi_i^k \xi_j^k$, $B_{ij} \sim \mathrm{Be}(p), p \in (0,1)$

$$H_N^{\text{que}}(\sigma) = -\frac{1}{N} \sum_{1\le i<j\le N} A_{ij}B_{ij}\sigma_i\sigma_j - h \sum_{1\le i\le N} \sigma_i, \qquad \sigma \in \{-1,1\}^N.$$

The annealed model

Notation: $\mathbb{P}_B[\,\cdot\,] = \mathbb{P}[\,\cdot \mid \mathcal{F}_{A\cup P}]$.

Then the Hamiltonian of the *(partially) annealed model* is

$$H_N^{\mathsf{ann}}(\sigma) = \mathbb{E}_B\left[H_N^{\mathsf{que}}(\sigma)\right] = -\frac{1}{N}\sum_{1\le i<j\le N} A_{ij}\,\mathbb{E}_B[B_{ij}]\sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i$$

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Examples:

- Ising on Erdős–Rényi, $\xrightarrow{\text{annealed}}$ Curie-Weiss

  [Bovier, Marello, P. (2021), den Hollander, Jovanovski (2021), BdHMPS (2022+)]

- Ising on Chung-Lu random graph: $B_{ij} \sim \mathrm{Be}(V_iV_j)$, V_i iid

  $\xrightarrow{\text{annealed}}$ Ising with product disorder: $B_{ij} = V_iV_j$, V_i iid

  [BdHMSP (2022+), Bovier, den Hollander, Marello (2022)]

- Dilute Hopfield $\xrightarrow{\text{annealed}}$ Hopfield

  [BdHMSP (2022+), an der Heiden (2007)]

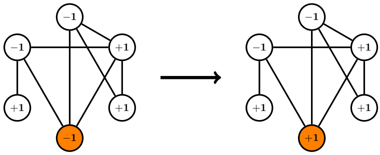
The Glauber dynamics

At equilibrium we define the Gibbs measure, $\sigma \in \mathcal{S}_N$,

$$\mu_{N,\beta}(\sigma) = \frac{e^{-\beta H_N(\sigma)}}{Z_{N,\beta}} \qquad \text{with} \qquad Z_{N,\beta} = \sum_{\sigma \in \mathcal{S}_N} e^{-\beta H_N(\sigma)}$$

were $\beta \in (0, \infty)$ is the inverse temperature and $Z_{N,\beta}$ the partition function. Continuous-time Glauber dynamics on $\mathcal{S}_N$ with Metropolis transition rates

$$p_N(\sigma, \sigma') = \begin{cases} \exp(-\beta[H_N(\sigma') - H_N(\sigma)]_+) & \text{if } \sigma \sim \sigma', \\ 0 & \text{otherwise.} \end{cases}$$



$\mu_{N,\beta}$ is the unique invariant and reversible measure.

Key questions

Assume metastability of the annealed model.
Questions:

1. Is the random model metastable too?

   In particular, denoting by P the law of the Markov chain, let $\tau_{\mathcal{A}}$ be the first return time to $\mathcal{A}$ and define last exit-biased distribution

$$\nu_{\mathcal{A},\mathcal{B}}(\sigma) \; = \; \frac{\mu(\sigma)\,\mathrm{P}_\sigma\big[\tau_{\mathcal{B}} < \tau_{\mathcal{A}}\big]}{\sum_{\sigma\in\mathcal{A}}\mu(\sigma)\,\mathrm{P}_\sigma\big[\tau_{\mathcal{B}} < \tau_{\mathcal{A}}\big]}, \qquad \sigma \in \mathcal{A}.$$

2. What is the mean hitting time of the more stable set $\mathcal{B}$, i.e.

$$\mathrm{E}_{\nu_{\mathcal{A},\mathcal{B}}}\big[\tau_{\mathcal{B}}\big] \; ?$$

Two kind of results: tail behaviour of the distribution and sharp estimates of the moments. All results quenched in A_{ij} and P_{ij}.

Definition of metastability

Consider a simple case where $\mathcal{M} = \{m_1, ..., m_k\} \subset \mathcal{S}$ is a set of points, e.g. minima of F. Then a Markov process $X = \{X_t : t \geq 0\}$ is said to be ρ-metastable with respect to the set of metastable points $\mathcal{M}$ if

$$\frac{\max_{m \in \mathcal{M}} \mathbb{P}_m \left[\tau_{\mathcal{M} \setminus m} < \tau_m \right]}{\min_{A \subset \mathcal{S} \setminus \mathcal{M}} \mathbb{P}_{\mu|A} \left[\tau_{\mathcal{M}} < \tau_A \right]} \leq \rho \ll 1,$$

Definition from Schlichting and Slowik ('19). More classical definition in BdH.

Results: metastability for the random model

Assumption

For $\mathbb{P}$-a.e. realization of the r.v.'s, there exist $N_0 < \infty$ and $\forall N$ disjoint subsets $\mathcal{M}_{1,N}, \ldots, \mathcal{M}_{K,N}$ of $\mathcal{S}_N$ such that, for all $N \geq N_0$, the annealed model is $\tilde{\rho}_N := \mathrm{e}^{-c_1 N}$-metastable with respect to $\{\mathcal{M}_{1,N}, \ldots, \mathcal{M}_{K,N}\}$.

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Theorem 1 (BdHMPS)

Under the Assumption and for $\mathbb{P}$-a.e. realization of the r.v.'s and any $c_2 \in (0, c_1)$, there exists $N_1(c_2) < \infty$ such that, for all $N \geq N_1(c_2)$ the random model is $\rho_N := \mathrm{e}^{-c_2 N}$-metastable with respect to $\mathcal{M} = \{\mathcal{M}_{1,N}, \ldots, \mathcal{M}_{K,N}\}$.

Results: tail estimates

Fix $i \in \{2, \ldots, K\}$ and take $\mathcal{A} = \mathcal{M}_{i,N}$. Choose $\mathcal{B}$ as the union of all the other metastable sets with lower free energy.
Notation: all quantities which refer to the annealed model have the tilda.

Theorem 2 (BdHMPS)

Under the Assumption, for any $s > 0$ and $\mathbb{P}$-a.e. realization of the r.v.'s

$$\lim_{N \to \infty} \mathbb{P}_B \left[\mathrm{e}^{-s-\alpha_N} \big(1 + o(1)\big) \leq \frac{\mathrm{E}_{\nu_{\mathcal{A},\mathcal{B}}} \left[\tau_{\mathcal{B}}\right]}{\tilde{\mathrm{E}}_{\tilde{\nu}_{\mathcal{A},\mathcal{B}}} \left[\tilde{\tau}_{\mathcal{B}}\right]} \leq \mathrm{e}^{s+2\alpha_N} \big(1 + o(1)\big) \right] \geq 1 - 4\mathrm{e}^{-\bar{c}s^2}$$

where $\bar{c} := 1/(\beta a\, b)^2$ and

$$\alpha_N := \frac{\beta^2}{2N^2} \sum_{1 \leq i < j \leq N} A_{ij}^2 \, \mathbb{V}\mathrm{ar}_B[B_{ij}].$$

Results: moments

Theorem 3 (BdHMPS)

For any $q \geq 1$ and $\mathbb{P}$-a.e. realization of the r.v.'s

$$\mathrm{e}^{-\alpha_N}\big(1+o(1)\big) \;\leq\; \frac{\mathbb{E}_B\left[\mathrm{E}_{\nu_{\mathcal{A},\mathcal{B}}}\left[\tau_{\mathcal{B}}\right]^q\right]^{1/q}}{\tilde{\mathrm{E}}_{\tilde{\nu}_{\mathcal{A},\mathcal{B}}}\left[\tilde{\tau}_{\mathcal{B}}\right]} \;\leq\; \mathrm{e}^{4q\alpha_N}\big(1+o(1)\big)$$

for all $N \geq N_1(c_2,\omega)$.

Strategy in two steps

Target result: tail estimates

$$\lim_{N\to\infty} \mathbb{P}_B \left[\mathrm{e}^{-s-\alpha_N}\big(1+o(1)\big) \le \frac{\mathrm{E}_{\nu_{\mathcal{A},\mathcal{B}}}[\tau_{\mathcal{B}}]}{\tilde{\mathrm{E}}_{\tilde{\nu}_{\mathcal{A},\mathcal{B}}}[\tilde{\tau}_{\mathcal{B}}]} \le \mathrm{e}^{s+2\alpha_N}\big(1+o(1)\big) \right] \;\ge\; 1-4\mathrm{e}^{-\bar{c}s^2}$$

First step: concentration inequality

$$\mathbb{P}_B \left[\left| \log \mathrm{E}_{\nu_{\mathcal{A},\mathcal{B}}}[\tau_{\mathcal{B}}] - \mathbb{E}_B\left[\log \mathrm{E}_{\nu_{\mathcal{A},\mathcal{B}}}[\tau_{\mathcal{B}}]\right] \right| > s \right] \;\le\; 4\mathrm{e}^{-\bar{c}s^2} + o(1),$$

Second step: annealed estimates

$$-\alpha_N(1+o(1)) \;\le\; \mathbb{E}_B\left[\log \mathrm{E}_{\nu_{\mathcal{A},\mathcal{B}}}[\tau_{\mathcal{B}}]\right] - \log \tilde{\mathrm{E}}_{\tilde{\nu}_{\mathcal{A},\mathcal{B}}}[\tilde{\tau}_{\mathcal{B}}] \;\le\; 2\alpha_N(1+o(1)),$$

Potential theoretic approach (Bovier, Eckhoff, Gayrard and Klein, 2001)

Translates the problem of understanding the metastable behaviour of Markov processes to the study of capacities of electric networks. Link between **mean metastable crossover time** and **capacity**.
For $\mathcal{A}, \mathcal{B}$ disjoint subsets of $\mathcal{S}_N$, the key formula is

$$\mathrm{E}_{\nu_{\mathcal{A},\mathcal{B}}}[\tau_{\mathcal{B}}] = \sum_{\sigma \in \mathcal{A}} \nu_{\mathcal{A},\mathcal{B}}(\sigma)\, \mathrm{E}_\sigma[\tau_{\mathcal{B}}] = \frac{1}{\mathrm{cap}(\mathcal{A},\mathcal{B})} \sum_{\sigma' \in \mathcal{S}_N} \mu(\sigma') h_{\mathcal{A}\mathcal{B}}(\sigma'),$$

where

$$\mathrm{cap}(\mathcal{A},\mathcal{B}) = \sum_{\sigma \in \mathcal{A}} \mu_N(\sigma)\, \mathrm{P}_\sigma(\tau_{\mathcal{B}} < \tau_{\mathcal{A}})$$

and $h_{\mathcal{A}\mathcal{B}}$ is called *harmonic function*

$$h_{\mathcal{A}\mathcal{B}}(\sigma) = \begin{cases} \mathrm{P}_\sigma(\tau_{\mathcal{A}} < \tau_{\mathcal{B}}) & \sigma \in \mathcal{S}_N \setminus (\mathcal{A} \cup \mathcal{B}), \\ \mathbb{1}_{\mathcal{A}}(\sigma) & \sigma \in \mathcal{A} \cup \mathcal{B}. \end{cases}$$

$\Rightarrow$ we need to prove the two steps (concentration and annealed estimates) for $\mathrm{cap}(\mathcal{A},\mathcal{B})$ and $h_{\mathcal{A}\mathcal{B}}$.

One related problem

Metastability for the dilute Curie-Weiss Potts model: spins in $\{1, ..., q\}$ and random couplings (together with Johan Dubbeldam, Vicente Lenz and Martin Slowik)

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Thank you for your attention!

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$\Rightarrow$ we need to prove the two steps (concentration and annealed estimates) for $\mathrm{cap}(\mathcal{A},\mathcal{B})$ and $h_{\mathcal{A}\mathcal{B}}$.

Variational principles for the capacity

Dirichlet principle

$$\mathrm{cap}(\mathcal{A},\mathcal{B}) = \inf_{g\in\mathcal{H}_{\mathcal{A}\mathcal{B}}} \frac{1}{2} \sum_{\sigma,\sigma'\in\mathcal{S}_N} \mu_N(\sigma) p_N(\sigma,\sigma')[g(\sigma)-g(\sigma')]^2.$$

$$\mathcal{H}_{\mathcal{A}\mathcal{B}} := \{g : \mathcal{S}_N \to [0,1] : g|_{\mathcal{A}} = 1, g|_{\mathcal{B}} = 0\}$$

Thomson principle

$$\mathrm{cap}(\mathcal{A},\mathcal{B}) = \sup_{\phi\in\mathcal{U}_{\mathcal{A}\mathcal{B}}} \frac{1}{\mathcal{D}(\phi)},$$

where

$$\mathcal{D}(\phi) = \sum_{(\sigma,\sigma')\in E} \frac{\phi(\sigma,\sigma')^2}{\mu_N(\sigma)p_N(\sigma,\sigma')}$$

$\mathcal{U}_{\mathcal{A}\mathcal{B}}$ is the space of all unit flows from $\mathcal{A}$ to $\mathcal{B}$

McDiarmid concentration inequality

Classical concentration inequality for functionals of independent random variables satisfying a bounded difference estimate

Theorem 4 (McDiarmid's inequality)

Let $\mathcal{X}$ be a Polish space. Consider a vector $X = (X_1, \dots, X_n)$ of independent $\mathcal{X}$-valued random variables and suppose that $f\colon \mathcal{X}^n \to \mathbb{R}$ satisfies, for any $i \in \{1, \dots, n\}$, the bounded differences inequality, i.e.

$$\big|f(X_1, \dots, X_n) - f(X_1, \dots, X_{i-1}, X_i', X_{i+1}, \dots, X_n)\big| \;\leq\; c_i \in [0, \infty) \quad \mathbb{P}\text{-a.s.},$$

where $(X_1', \dots, X_n')$ is an independent copy of $(X_1, \dots, X_n)$. Then, $\mathbb{P}$-a.s. for any $t \geq 0$,

$$\mathbb{P}\big[f(X) - \mathbb{E}[f(X)] > t\big] \;\leq\; \mathrm{e}^{-t^2/2v},$$

where $v := \frac{1}{4}\sum_{i=1}^{n} c_i^2$.

Concentration inequality for the capacity

Consider two disjoint subsets $\mathcal{X}, \mathcal{Y} \subset \mathcal{S}_N$. Then, $\mathbb{P}$-a.s., for any $t \geq 0$,

$$\mathbb{P}_B\Big[\Big|\log\big(Z\mathrm{cap}(\mathcal{X},\mathcal{Y})\big) - \mathbb{E}_B\Big[\log\big(Z\mathrm{cap}(\mathcal{X},\mathcal{Y})\big)\Big]\Big| > t\Big] \;\leq\; 2\,\mathrm{e}^{-2t^2\bar{c}},$$

where $\bar{c} = 1/(\beta a\, b)^2$.

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where $\bar{c} = 1/(\beta a\, b)^2$.

Proof: the map

$$(B_{ij}) \;\longmapsto\; F\big((B_{ij})\big) := \log\big(Z^B \mathrm{cap}^B(\mathcal{X},\mathcal{Y})\big)$$

satisfies a bounded difference inequality:

$$\big|F\big((B_{ij})\big) - F\big((B'_{ij})\big)\big| \;\leq\; \frac{\beta ab}{N} \qquad \mathbb{P}\text{-a.s.},$$

when $B'_{ij} = B_{ij}$ for any $(i,j) \neq (k,\ell)$ and $B'_{k\ell}$ independent copy of $B_{k\ell}$.

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where $\bar{c} = 1/(\beta a\, b)^2$.

Proof: the map

$$(B_{ij}) \;\longmapsto\; F\big((B_{ij})\big) := \log\big(Z^B \mathrm{cap}^B(\mathcal{X},\mathcal{Y})\big)$$

satisfies a bounded difference inequality:

$$\big|F\big((B_{ij})\big) - F\big((B'_{ij})\big)\big| \;\leq\; \frac{\beta ab}{N} \qquad \mathbb{P}\text{-a.s.},$$

when $B'_{ij} = B_{ij}$ for any $(i,j) \neq (k,\ell)$ and $B'_{k\ell}$ independent copy of $B_{k\ell}$.

$$Z^B \mathrm{cap}^B(\mathcal{X},\mathcal{Y}) = \inf_f \frac{Z}{2} \sum_{\sigma,\eta\in\mathcal{S}_N} \mathrm{e}^{-\beta\left(H^B(\sigma)\vee H^B(\eta)\right)} \left(f(\sigma) - f(\eta)\right)^2$$

Concentration inequality for the capacity

Consider two disjoint subsets $\mathcal{X}, \mathcal{Y} \subset \mathcal{S}_N$. Then, $\mathbb{P}$-a.s., for any $t \geq 0$,

$$\mathbb{P}_B\Big[\Big|\log\big(Z\mathrm{cap}(\mathcal{X},\mathcal{Y})\big) - \mathbb{E}_B\Big[\log\big(Z\mathrm{cap}(\mathcal{X},\mathcal{Y})\big)\Big]\Big| > t\Big] \;\leq\; 2\,\mathrm{e}^{-2t^2\bar{c}},$$

where $\bar{c} = 1/(\beta a\, b)^2$.

Proof: the map

$$(B_{ij}) \;\longmapsto\; F\big((B_{ij})\big) := \log\big(Z^B \mathrm{cap}^B(\mathcal{X},\mathcal{Y})\big)$$

satisfies a bounded difference inequality:

$$\big|F\big((B_{ij})\big) - F\big((B'_{ij})\big)\big| \;\leq\; \frac{\beta ab}{N} \qquad \mathbb{P}\text{-a.s..}$$

Obtain BDI via: Dirichlet principle, comparison of Dirichlet forms and

$$\max_{\sigma\in\mathcal{S}_N}\big|H^B(\sigma) - H^{B'}(\sigma)\big| \leq |A_{k\ell}|\frac{|B_{kl} - B'_{kl}|}{N} \leq \frac{a\,b}{N}$$

Annealed estimates for the capacity

Consider two disjoint subsets $\mathcal{X}, \mathcal{Y} \subset \mathcal{S}_N$. Then, $\mathbb{P}$-a.s.,

$$\Big|\mathbb{E}_B\big[\log\big(Z\mathrm{cap}(\mathcal{X},\mathcal{Y})\big)\big] - \log\big(\tilde{Z}\,\widetilde{\mathrm{cap}}(\mathcal{X},\mathcal{Y})\big)\Big| \;=\; \alpha_N + O\Big(\frac{1}{\sqrt{N}}\Big)$$

Annealed estimates for the capacity

Consider two disjoint subsets $\mathcal{X}, \mathcal{Y} \subset \mathcal{S}_N$. Then, $\mathbb{P}$-a.s.,

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Via comparison of the Dirichlet form for functions $\mathcal{E}_N(f)$ and the Dirichlet form for unit-flows $\mathcal{D}_N(\varphi)$

$$\begin{aligned} \mathbb{E}_B\big[Z_N\mathcal{E}_N(f)\big] \;&=\; \tilde{Z}_N\tilde{\mathcal{E}}_N(f)\,\mathrm{e}^{\alpha_N}\big(1+O(N^{-1/2})\big) && \forall\, f\in\mathcal{H}_{\mathcal{X},\mathcal{Y}},\\ \mathbb{E}_B\big[Z_N^{-1}\mathcal{D}_N(\varphi)\big] \;&=\; \tilde{Z}_N^{-1}\tilde{\mathcal{D}}_N(\varphi)\,\mathrm{e}^{\alpha_N}\big(1+O(N^{-1/2})\big) && \forall\,\varphi\in\mathcal{U}_{\mathcal{X},\mathcal{Y}}. \end{aligned}$$

And via annealed estimates on $\Delta_N(\sigma) := H_N(\sigma) - \tilde{H}_N(\sigma)$, i.e. for any $\sigma \in \mathcal{S}_N$ and $\mathbb{P}$-a.s.,

$$\mathbb{E}_B\Big[\mathrm{e}^{\pm\beta\Delta_N(\sigma)}\Big] \;=\; \mathrm{e}^{\alpha_N}\big(1+O(N^{-1})\big).$$