Size-biased diffusion limits for the inclusion process

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April 4, 2023

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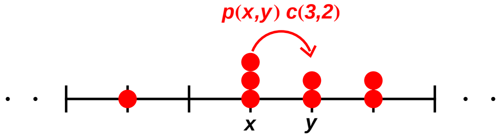
The inclusion process



Lattice: Λ of size $|\Lambda| = L$

State space: $E_L = \{0, 1, ..\}^\Lambda$

$\boldsymbol{\eta} = (\eta_x)_{x\in\Lambda}$

Jump rates: $p(x,y)\, c(\eta_x, \eta_y)$ with $c(k,l) = 0 \;\Leftrightarrow\; k = 0$

p irreducible and **homogeneous** $\sum_{y\in\Lambda} \big(p(x,y) - p(y,x)\big) = 0\ ,\quad x \in \Lambda$

Generator:

$$\mathcal{L}f(\boldsymbol{\eta}) = \sum_{x,y\in\Lambda} p(x,y)c(\eta_x,\eta_y)\big(f(\boldsymbol{\eta}^{x,y}) - f(\boldsymbol{\eta})\big)$$

Inclusion process $c(\eta_x, \eta_y) = \eta_x\eta_y + d\eta_x\ ,\quad d \geq 0$

[Giardiná et al. (2009); Waclaw, Evans (2012); Cocozza-Thivent (1985)]

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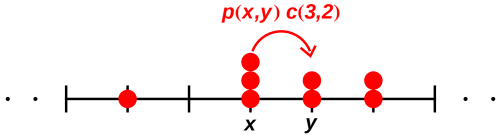

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Inclusion process $c(\eta_x, \eta_y) = \eta_x\eta_y + d\eta_x\ ,\quad d \geq 0$

Explosive condensation model $c(\eta_x, \eta_y) = \eta_x^\gamma(d + \eta_y^\gamma)\ ,\quad d \geq 0\ , \gamma > 0$

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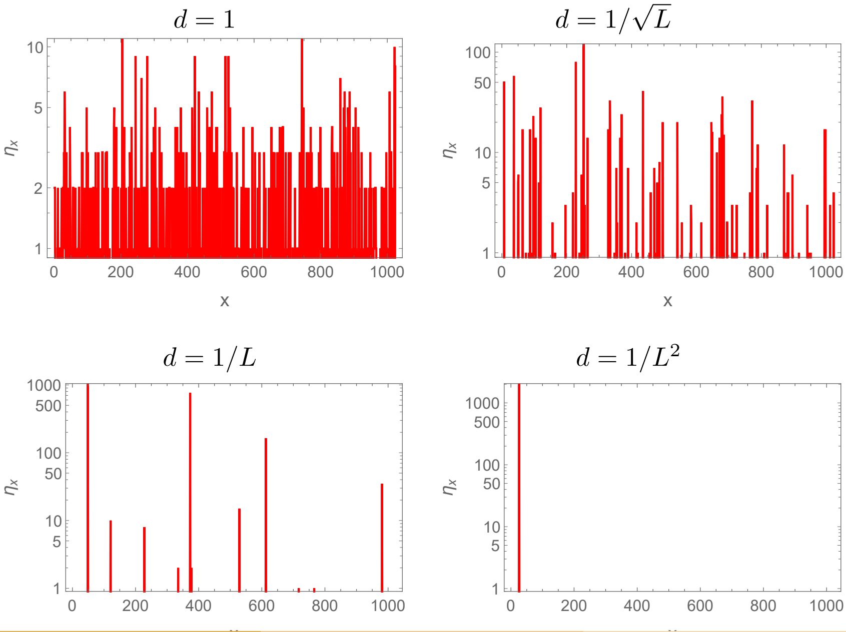
Condensation in the inclusion process



Outline

- Stationary results for spatially homogeneous IP and related models

  [W. Jatuviriyapornchai, P. Chleboun, S. G., Structure of the condensed phase in the inclusion process, JSP 178, 682-710 (2020)]

  [P. Chleboun, S. Gabriel, S. G., Poisson-Dirichlet asymptotics in condensing particle systems, EJP 27, 1-35 (2022)]

- Diffusion limits for IP on a complete graph

  [P. Chleboun, S. Gabriel, S. G., in preparation]

Previous results

- Metastable stationary dynamics of a single condensate
  [G., Redig, Vafayi (2013)], [Bianchi, Dommers, Giardiná (2017); Kim, Seo (2021)]
- Mean field rate equations for the bulk [Jatuviriyapornchai, G. (2019)]
- Hydrodynamic limit for cluster dynamics via dualtiy [Ayala, Carinci, Redig (2021)]

Stationary measures

canonical measures are Dirichlet multinomials

$$\pi_{L,N}[d\eta] = \frac{\mathbb{1}_{X_{L,N}}(\eta)}{Z_{L,N}} \prod_{x\in\Lambda} w(\eta_x)\, d\eta$$

where $w(n) = \frac{\Gamma(n+d)}{n!\Gamma(d)} \simeq d\, n^{d-1}$ and $Z_{L,N} = \frac{\Gamma(N+dL)}{N!\Gamma(dL)}$

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order statistics $\hat{\eta} := \big(\eta_{(L)}, \ldots, \eta_{(1)}\big)$ for $\eta \in X_{L,N}$

size-biased sample $\tilde{\eta} := \big(\eta_{\sigma(1)}, \ldots, \eta_{\sigma(L)}\big)$

$$\sigma(1) = x \in \Lambda \text{ w.prob. } \frac{\eta_x}{N}\ , \quad \sigma(2) = y \in \Lambda \setminus \sigma(1) \text{ w.prob. } \frac{\eta_y}{N - \eta_{\sigma(1)}}\ \ldots$$

partitions $\frac{1}{N}\tilde{\eta} \in \Delta := \Big\{(q_1, q_2, \ldots) : q_k \geq 0 : \sum_k q_k = 1\Big\}$

$$\frac{1}{N}\hat{\eta} \in \nabla := \Big\{(q_1, q_2, \ldots) : q_1 \geq q_2 \geq \ldots \geq 0 : \sum_k q_k = 1\Big\}$$

Condensation in IP

Asymptotic behaviour

IP $(\pi_{L,N})_{L,N}$ exhibits a CT with $\rho_c = 0$ as $N/L \to \rho, d = d_L \to 0$ and

- $(\eta_1, \ldots, \eta_k) \xrightarrow{D} (0, \ldots, 0)$ for all fixed $k \geq 1$
- $dL \to \infty$: $d\big(\tilde{\eta}_1, \ldots, \tilde{\eta}_k\big) \xrightarrow{D} i.i.d.\ \mathrm{Exp}(1/\rho)$ for all fixed $k \geq 1$
- $dL \to \theta \in [0, \infty)$: $\frac{1}{N}\tilde{\eta} \xrightarrow{D} \mathrm{GEM}(\theta)$ or $\frac{1}{N}\hat{\eta} \xrightarrow{D} \mathrm{PD}(\theta)$
- $dL \log L \to 0$: complete condensation with $N - \eta_{(L)} \xrightarrow{D} 0$

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Let $U_1, U_2, \ldots \sim$ Beta$(1, \theta)$ iidrvs on $[0, 1]$ with PDF $\theta(1 - x)^{\theta - 1}$.

A random partition $V = (V_k : k \in \mathbb{N}) \in \Delta$ is $\mathrm{GEM}(\theta)$ distributed if

$$V_1 = U_1\ , \quad V_2 = (1 - U_1)U_2\ , \quad V_3 = (1 - U_1)(1 - U_2)U_3\ , \ \ldots$$

Then the order statistics $\nabla \ni \hat V \sim \mathrm{PD}(\theta)$ have Poisson-Dirichlet distribution.

[Kingman (1975), Griffiths (1980), Engen (1978), McCloskey (1965)]

IP on the complete graph

Generator $\mathcal{L}f(\eta) = \sum_{x,y\in\Lambda}(\eta_x\eta_y + d_L\eta_x)\big(f(\eta^{xy}) - f(\eta)\big)$, $\eta \in E_{L,N}$

Particle positions $\sigma = (\sigma_i : i = 1, \ldots, N) \in \Lambda^N$, IP $(\sigma(t) : t \geq 0)$

$$\mathfrak{L}g(\sigma) = \sum_{i,j=1}^{N} \big(g(\sigma^{i\to\sigma_j}) - g(\sigma)\big) + d_L \sum_{i=1}^{N}\sum_{x\in\Lambda} \big(g(\sigma^{i\to x}) - g(\sigma)\big)$$

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Empirical measure $\mu^\sigma := \frac{1}{N}\sum_{i=1}^{N} \delta_{\frac{\sigma_i}{L}} \in \mathcal{M}_1([0,1])$ with $\mu^\sigma(h) = \frac{1}{N}\sum_{i=1}^{N} h(\frac{\sigma_i}{L})$

$h \in C([0,1])$

$$\mathfrak{L}\mu^\sigma(h) = \underbrace{\frac{1}{N}\sum_{i,j=1}^{N}\Big[h\big(\tfrac{\sigma_j}{L}\big) - h\big(\tfrac{\sigma_i}{L}\big)\Big]}_{=0} + \underbrace{\frac{1}{N}\sum_{i=1}^{N}\frac{d_L L}{L}\sum_{x\in\Lambda}\Big[h\big(\tfrac{x}{L}\big) - h\big(\tfrac{\sigma_i}{L}\big)\Big]}_{\to\mu^\sigma(\mathfrak{A}h)}$$

with **mutation operator** $\mathfrak{A}h(v) = \theta\int_0^1 \big[h(u) - h(v)\big]du$

Fleming-Viot process on type space

$H(\mu^\sigma) = \prod_{k=1}^n \mu^\sigma(h_k)$, $h_k \in C([0,1])$, $n \in \mathbb{N}$ is a **core** for $\mathfrak{L}$

$$\mathfrak{L} \prod_{k=1}^n \mu^\sigma(h_k) = \sum_{k,l=1}^n \big(\mu^\sigma(h_k h_l) - \mu^\sigma(h_k)\mu^\sigma(h_l)\big) \prod_{m\neq k,l} \mu^\sigma(h_m)$$

$$+ \sum_{k=1}^n \mu^\sigma(\mathfrak{A} h_k) \prod_{m\neq k} \mu(h_m) + o(1)$$

thus $\big(\mu^{\sigma(t)} : t \geq 0\big) \xrightarrow{D} (\mu_t : t \geq 0)$ **Fleming-Viot process** with generator

$$\mathfrak{G} \prod_{k=1}^n \mu(h_k) = \sum_{k,l=1}^n \big(\mu(h_k h_l) - \mu(h_k)\mu(h_l)\big) \prod_{m\neq k,l} \mu(h_m) + \sum_{k=1}^n \mu(\mathfrak{A} h_k) \prod_{m\neq k} \mu(h_m)$$

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measure-valued diffusion $d\mu_t(h) = \mu_t(\mathfrak{A} h)dt + dM_t(h)$

with martingale $M_t(h)$ with QV $\int_0^t \Gamma \mu_s(h) ds = 4 \int_0^t \big(\mu_s(h^2) - \mu_s(h)^2\big) ds$

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and $\mathfrak{A}h(v) = \theta\int_0^1 \big[h(u) - h(v)\big]du$

[Ethier, Kurtz (1993)]

is equivalent to **Poisson-Dirichlet diffusion** $\big(q(t) : t \geq 0\big)$ on ∇ with generator

$$\mathcal{L}_{PD}f(q) = \sum_{i,j=1}^{\infty} q_i q_j (\partial_{q_i} - \partial_{q_j})^2 f(q) - \theta\sum_{i=1}^{\infty} q_i\partial_{q_i} f(q)$$

defined on a **core** $1, \phi_2, \phi_3 \ldots$ with $\phi_m(q) = \sum_i q_i^m$ [Ethier, Kurtz (1981)]

The partition $q(t) \in \nabla$ corresponds to the ordered **atoms** of μ_t.

[Griffiths, Ruggerio, Spanò, Zhou (2021)]

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$$\frac{1}{d_L L}\mathcal{L}_{PD} f(q) = \frac{1}{d_L L}\sum_{i,j=1}^{\infty} q_i q_j (\partial_{q_i} - \partial_{q_j})^2 f(q) - \sum_{i=1}^{\infty} q_i \partial_{q_i} f(q)$$

after reaching a single condensate with $\mu_t = \delta_z$, $z \in [0,1]$

$$dM_t(h) \equiv 0 \quad \text{and} \quad \frac{d\mu_t(h)}{dt} = \delta_z(\mathfrak{A}h) = \int_0^1 \big[h(u) - h(z)\big]du$$

metastability uniform jumps of the condensate at (rescaled) rate 1

Measure-valued process $d_L L \to \theta$

Generator $\mathcal{L}f(\eta) = \sum_{x,y\in\Lambda} (\eta_x \eta_y + d_L \eta_x)\big(f(\eta^{xy}) - f(\eta)\big)\ ,\quad \eta \in E_{L,N}$

Empirical measure on mass space $\mu^\eta := \sum_{x\in\Lambda} \frac{\eta_x}{N} \delta_{\frac{\eta_x}{N}} \in \mathcal{M}_1([0,1])$

$$\mu^\eta(h) = \sum_{x\in\Lambda} \tfrac{\eta_x}{N} h\big(\tfrac{\eta_x}{N}\big) = \sum_{x\in\Lambda} \tilde{h}\big(\tfrac{\eta_x}{N}\big)\ ,\quad h \in C([0,1])$$

state space $E := \mu^{(\cdot)}(\overline{\nabla}) \subsetneq \mathcal{M}_1([0,1])$

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Lemma

The closure of $(\mathcal{G}, \mathcal{D}_\mathcal{G})$ with

$$\mathcal{G} \prod_{k=1}^n \mu(h_k) = \sum_{k,l=1}^n \big(\mu(\tilde{h}_k' \tilde{h}_l') - \mu(\tilde{h}_k')\mu(\tilde{h}_l')\big) \prod_{m\neq k,l} \mu(h_m) + \sum_{k=1}^n \mu(\mathcal{A}h_k) \prod_{m\neq k} \mu(h_m)$$

domain $\mathcal{D}_\mathcal{G} =$ sub-algebra of $C(E)$ generated by $\mu \mapsto \mu(h)$, $h \in C^3([0,1])$

generates a Feller process on E, where $\tilde{h}'(z) = \big(zh(z)\big)' = h(z) + zh'(z)$ and

$$\mathcal{A}h(z) = z(1-z)h''(z) + \big(2 - (2+\theta)z\big)h'(z) + \theta\big(h(0) - h(z)\big) \ .$$

Measure-valued process $d_L L \to \theta$

Theorem

Let $\mu^{\eta(0)} \xrightarrow{D} \mu_0 \in E$. Then for all $\rho > 0$ as $N/L \to \rho$, $d_L L \to \theta \geq 0$

$$\big(\mu^{\eta(t)} : t \geq 0\big) \xrightarrow{D} (\mu_t : t \geq 0) \qquad \text{on } D\big([0,\infty), E\big)$$

where $(\mu_t : t \geq 0)$ is a **measure-valued process** with generator $\mathcal{G}$.

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- **measure-valued diffusion** $d\mu_t(h) = \mu_t(\mathcal{A}h)dt + d\mathcal{M}_t(h)$
- mass is conserved $\big(h(z) \equiv 1\big)$, δ_0 describes **mass below macro. scale**
  $h(z) = z$ describes second moment of the mass partition
- Let $(Z_t : t \geq 0)$ be the process on $[0,1]$ with generator $\mathcal{A}$, then we have

  the **duality** $\mathbb{E}_{\mu_0}\big[\mu_t(h)\big] = \mathbf{E}_{\mu_0}\big[h(Z_t)\big]$ for all $t \geq 0$.

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  the **duality** $\quad \mathbb{E}_{\mu_0}\big[\mu_t(h)\big] = \mathbf{E}_{\mu_0}\big[h(Z_t)\big]$ for all $t \geq 0$.
- Equivalence to PD diffusion $\big(q(t) : t \geq 0\big)$: $\quad (\mu_t : t \geq 0) \sim \big(\mu^{q(t)} : t \geq 0\big)$

Measure-valued process $d_L L \to \infty$

$$\mathcal{A}h(z) = z(1-z)h''(z) + \big(2 - (2 + d_L L)z\big)h'(z) + d_L L\big(h(0) - h(z)\big)$$

for intermediate scaling regimes $d_L L \to \infty$ we get

$$\frac{1}{d_L L}\mathcal{G}\prod_{k=1}^{n}\mu(h_k) \to \sum_{k=1}^{n}\mu\big(h(0) - h(z) - zh'(z)\big)\prod_{m\neq k}\mu(h_m)\ ,$$

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i.e. $d\mu_t(h) = \mu_t\big(h(0) - h(z) - zh'(z)\big)dt$ and $\mu_t(h) = \mathbf{E}_{\mu_0}\big[h(Z_t)\big]$

and macroscopic clusters disappear with rate $d_L L$

Measure-valued process $d_L L \to \infty$

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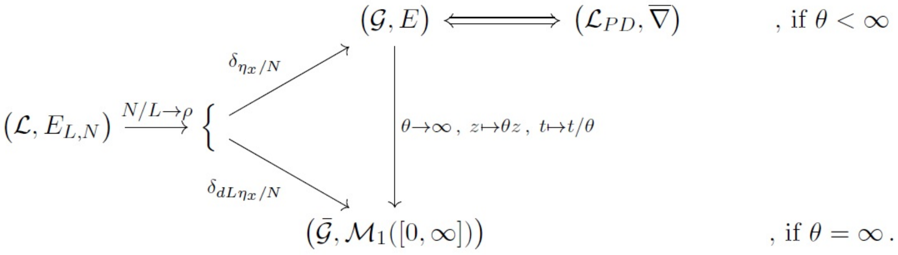

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Empirical measure on mass scale ρ/d_L: $\bar{\mu}^\eta := \sum_{x\in\Lambda} \frac{\eta_x}{N} \delta_{d_L L \frac{\eta_x}{N}} \in \mathcal{M}_1([0,\infty))$

$$\frac{1}{dL}\mathcal{L} \prod_{k=1}^{n} \bar{\mu}^\eta(h_k) = \sum_{k=1}^{n} \bar{\mu}^\eta(\bar{\mathcal{A}} h_k) \prod_{m\neq k} \bar{\mu}^\eta(h_m) + o(1) \ ,$$

where $\bar{\mathcal{A}} h(z) = z h''(z) + (2-z) h'(z) + (h(0) - h(z))$, $z \in [0,\infty)$

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where $\bar{\mathcal{A}} h(z) = z h''(z) + (2-z)h'(z) + (h(0) - h(z)) \ , \quad z \in [0,\infty)$

Theorem

Let $\mu^{\eta(0)} \xrightarrow{D} \mu_0 \in \mathcal{M}([0,\infty])$. Then for all $\rho > 0$ as $N/L \to \rho$, $d_L L \to \infty$

$$\big(\bar{\mu}^{\eta(t/(d_L L))} : t \geq 0\big) \xrightarrow{D} (\bar{\mu}_t : t \geq 0) \qquad \text{on } D\big([0,\infty), \mathcal{M}([0,\infty])\big)$$

where $(\mu_t : t \geq 0)$ is a **measure-valued process** with generator

$$\bar{\mathcal{G}} \prod_{k=1}^{n} \bar{\mu}(h_k) = \sum_{k=1}^{n} \bar{\mu}(\bar{\mathcal{A}} h_k) \prod_{m\neq k} \bar{\mu}(h_m) \ , \quad h_k \in C_c^3([0,\infty]) \cap \text{constants}$$

Measure-valued process $d_L L \to \infty$

- **measure-valued process** $d\bar{\mu}_t(h) = \mu_t(\bar{\mathcal{A}}h)dt$ and $\bar{\mu}_t(h) = \mathbf{E}_{\mu_0}\big[h(\bar{Z}_t)\big]$
- **Duality** $\bar{\mu}_t = \mathrm{Law}(\bar{Z}_t)$ with $(\bar{Z}_t : t \geq 0)$ on $[0, \infty)$ with generator $\bar{\mathcal{A}}$

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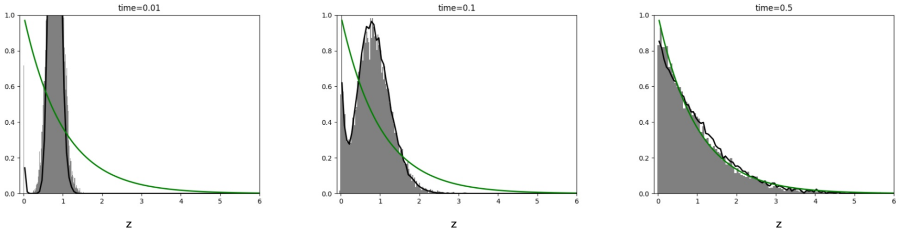
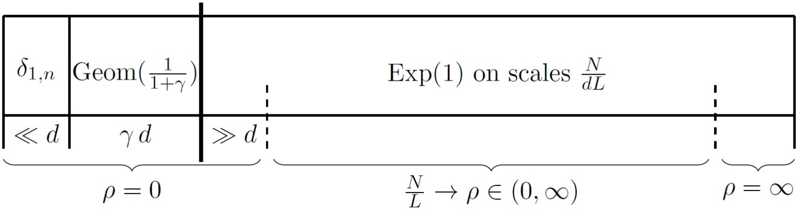


jump diffusion $\bar{\mathcal{A}}$ (histogram) , inclusion process $\mathcal{L}$ (black)

$N = L = 1024,\ d_L = L^{-1/2} = \frac{1}{32}$

Discussion

- Limiting dynamics of the condensate for all scalings of d_L
- Results are independent of ρ, can be extended to $\rho \in \{0, \infty\}$
  as long as $L \to \infty$ and $dL/N \to 0$



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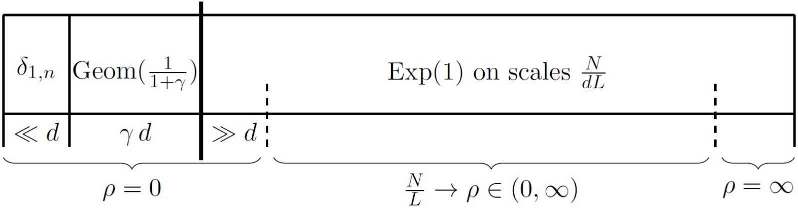

- work in progress for generalized kernels and ZRP with non-trivial bulk
- extend $d_L L \to \infty$ case to dense random graphs?

The end

Thank you!

Generalized IP-like models

Consider a system with $\pi_{L,N}[d\eta] = \frac{\mathbb{1}_{X_{L,N}}(\eta)}{Z_{L,N}} \prod_{x\in\Lambda} w_L(\eta_x)\, d\eta$ and

$$(A1) \qquad \|w_L - w\|_\infty \to 0 \quad \text{where wlog} \quad \sum_{n=0}^{\infty} w(n) = 1\ , \quad w(0) > 0\ ,$$

and $\sup_n \big| w(n-1) \wedge w(n) \big| > 0$ or $w(0) = 1$, as well as

$$(A2) \qquad \lim_{J\to\infty} \lim_{L\to\infty} \sup_{n>J} \big| n w_L(n) L - \theta \big| = 0\ .$$

Theorem [CGG (2022)]

As $L, N \to \infty$, $N/L \to \rho$ the system exhibits a **condensation transition** with

$$\rho_c = \sum_{n=0}^{\infty} n w(n) \in [0, \infty) \quad \text{and } \textbf{background density } \rho_c \text{ for } \rho > \rho_c\ .$$

The condensed mass fraction $\alpha = \alpha(\rho) = (\rho - \rho_c)/\rho$ is distributed as

$$\pi_{L,N}\Big[\frac{1}{N}\hat\eta \in \cdot\ \Big] \xrightarrow{D} \mathrm{PD}_{[0,\alpha]}(\theta) \quad \text{as } L, N \to \infty,\ N/L \to \rho \geq \rho_c\ .$$

Generalized IP-like models

Proof. $\mathrm{PD}(\theta)$ is the unique reversible distribution of **split-merge dynamics** on ∇

$$\mathcal{G}_\theta f(q) = \sum_{i\neq j} q_i q_j \Big[f(\widehat{M}_{ij} q) - f(q) \Big] + \theta \sum_i q_i^2 \Big[\int_0^1 f(\widehat{S}_i^u q) du - f(q) \Big]$$

proven for $\theta \in [0, 1]$ [Zerner et al. (2004), Schramm (2005)]

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Consider a discrete approximation [Ioffe, Tóth (2020)]

$$\mathcal{G}_\theta^{N,\epsilon} f(q) = \tfrac{N}{N-1} \sum_{i\neq j} q_i q_j \mathbb{1}_{q_i, q_j \geq \epsilon} \Big[f(\widehat{M}_{ij} q) - f(q) \Big] + \tfrac{\theta}{N-1} \sum_i q_i \mathbb{1}_{q_i \geq 2\epsilon} \Big[\sum_{k=\epsilon N}^{N(q_i-\epsilon)} f(\widehat{S}_i^{k/(Nq_i)} q) du - f(q) \Big]$$

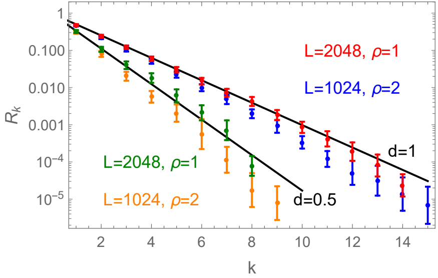
and show that $\mathcal{G}_\theta^{N,\epsilon} \to \mathcal{G}_\theta$ as $N \to \infty$, $\epsilon \to 0$ on $C_b(\overline{\nabla})$ and

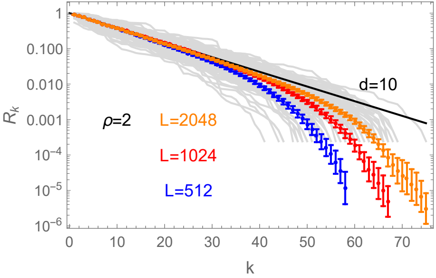
$$\Big| \pi_{L,N}\Big(f\big(\tfrac{1}{N}\hat{\eta}\big) \mathcal{G}_\theta^{N,\epsilon} g\big(\tfrac{1}{N}\hat{\eta}\big) \Big) - \pi_{L,N}\Big(g\big(\tfrac{1}{N}\hat{\eta}\big) \mathcal{G}_\theta^{N,\epsilon} f\big(\tfrac{1}{N}\hat{\eta}\big) \Big) \Big| \to 0$$

as $L, N \to \infty$, $N/L \to \rho \geq 0$.

GEM/PD regime for IP

$$dL \to \theta \in (0, \infty) \quad : \quad \frac{1}{N} \tilde{\eta} \xrightarrow{D} \mathrm{GEM}(\theta)$$



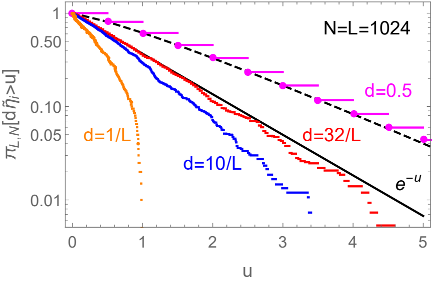


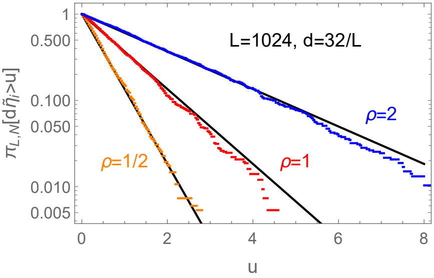
observe $R_k(\eta) = 1 - \frac{1}{N} \sum_{i=1}^{k} \tilde{\eta}_i$, then

$$\langle R_k \rangle_{L,N} \to \left(\frac{\theta}{1+\theta}\right)^k \quad \text{as } L, N \to \infty,\ N/L \to \rho,\ dL \to \theta\ .$$

Intermediate regime for IP

$$dL \to \infty \ : \quad d\big(\tilde{\eta}_1, \ldots, \tilde{\eta}_k\big) \xrightarrow{D} i.i.d.\ \mathrm{Exp}(1/\rho)$$





Condensation

canonical measures $\pi_{L,N}$ on $X_{L,N} = \{\boldsymbol{\eta} \in X_L : \sum_{x\in\Lambda} \eta_x = N\}$

thermodynamic limit $L, N \to \infty\ , \quad N/L \to \rho \geq 0$

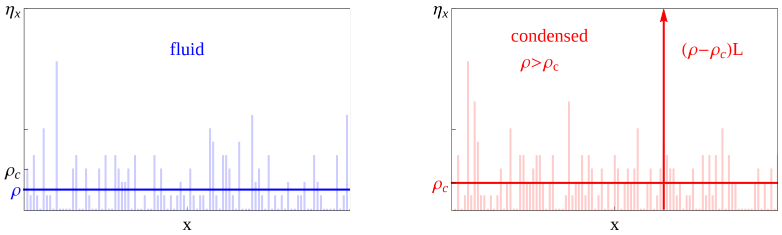
assume that $\nu_\rho := \lim_{L,N} \pi_{L,N}[\eta_x \in .]$ exists (and is unique)

background density $\rho_b := \langle \eta_x \rangle_\rho \leq \rho = \lim_{L,N} \langle \eta_x \rangle_{L,N}$

A hom. SPS exhibits condensation with bg dens. $\rho_b \geq 0$, if ν_ρ exists and $\rho_b < \rho$.

The SPS exhibits a condensation transition with critical density $\rho_c \geq 0$,

if ν_ρ exists for all $\rho \geq 0$, and $\rho_b \begin{cases} = \rho\ , & \rho < \rho_c \\ < \rho\ , & \rho > \rho_c \end{cases}$.



Condensation in IP

GEM/PD regime

Choose $\frac{n_1}{N} \to x_1$, $\frac{n_2}{N} \to (1-x_1)x_2$, $\ldots$, $\frac{n_k}{N} \to (1-x_1)\cdots(1-x_{k-1})x_k$

with $x_1, \ldots, x_k \in (0,1)$, $k \in \mathbb{N}$ fixed and show

$$N \cdots \Big(N - \sum_{i=1}^{k-1} n_i\Big)\, \pi_{L,N}[\tilde{\eta}_1..\tilde{\eta}_k = n_1..n_k] \to \theta^k \prod_{i=1}^{k} (1-x_i)^{\theta-1}$$

as $L \to \infty$, $N/L \to \rho$ for all $\rho > 0$.

Condensation in IP

GEM/PD regime

Choose $\frac{n_1}{N} \to x_1$, $\frac{n_2}{N} \to (1-x_1)x_2$, ..., $\frac{n_k}{N} \to (1-x_1)\cdots(1-x_{k-1})x_k$

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as $L \to \infty$, $N/L \to \rho$ for all $\rho > 0$.

Intermediate regime

Choose $\frac{n_i}{N} \to x_i > 0$, $k \in \mathbb{N}$ fixed and show

$$d^{-k}\, \pi_{L,N}[\tilde\eta_1..\tilde\eta_k = n_1..n_k] \to \frac{1}{\rho^k} \prod_{i=1}^{k} e^{-x_i/\rho}$$

as $L \to \infty$, $N/L \to \rho$ for all $\rho > 0$.