EINDHOVEN UNIVERSITY OF TECHNOLOGY

Metastability and condensation in inclusion processes

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Inclusion process on finite graph

Interacting particle system with N particles

Fixed vertex set S with $|S| < \infty$

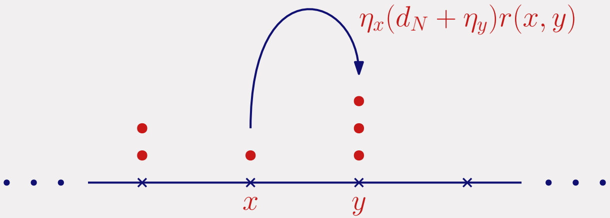
Configuration $\eta = (\eta_x)_{x \in S} \in \{0, \dots, N\}^S$, $\eta_x = \#$ particles on $x \in S$

Underlying random walk on S with transition rates $r(x, y)$

Inclusion process is continuous time Markov process with generator

$$\mathcal{L}f(\eta) = \sum_{x,y \in S} \eta_x(d_N + \eta_y) r(x,y) \left[f(\eta^{x,y}) - f(\eta) \right]$$

Particle jump rates



Particle jump rates can be split into

$\eta_x \, d_N \, r(x,y)$	independent random walkers	diffusion
$\eta_x \, \eta_y \, r(x,y)$	attractive interaction	inclusion

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Comparison with other processes

$\eta_x \, (1 - \eta_y) \, r(x,y)$	exclusion process
$g(\eta_x) \, r(x,y)$	zero range process

Reversible inclusion process

Irreducible random walk $r(\cdot,\cdot)$ reversible w.r.t. some measure $m(\cdot)$

$$m(x)\,r(x,y) = m(y)\,r(y,x) \qquad \forall\, x,y \in S$$

Normalized such that $\max_{x\in S} m(x) = 1$

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Then, also inclusion process reversible w.r.t. probability measure

$$\mu_N(\eta) = \frac{1}{Z_N}\prod_{x\in S} m(x)^{\eta_x}\, w_N(\eta_x)$$

where Z_N is a normalization constant and $w_N(k) = \dfrac{\Gamma(d_N+k)}{k!\,\Gamma(d_N)}$

Condensation

Let $S_\star = \{x \in S : m(x) = 1\}$ and $\eta^{x,N}$ the configuration η with $\eta_x = N$

Proposition

Suppose that $d_N \log N \to 0$ *as* $N \to \infty$. *Then*

$$\lim_{N\to\infty} \mu_N(\eta^{x,N}) = \frac{1}{|S_\star|} \qquad \forall x \in S_\star$$

Movement of the condensate

Consider the following process on $S_\star \cup \{0\}$:

$$X_N(t) = \sum_{x \in S_\star} x \, \mathbb{1}_{\{\eta_x(t)=N\}}$$

Theorem (Bianchi, D., Giardinà, 2017)

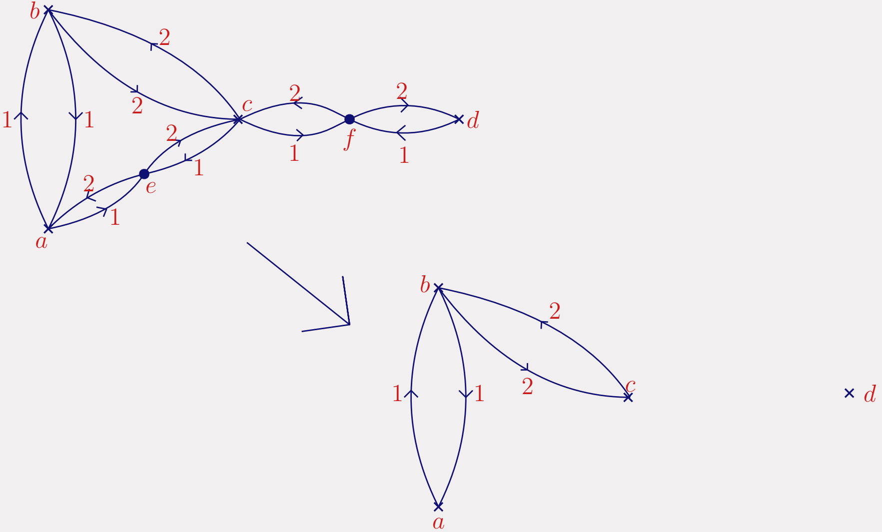
Suppose that $d_N \log N \to 0$ as $N \to \infty$ and that $\eta_y(0) = N$ for some $y \in S_\star$. Then

$$X_N(t/d_N) \text{ converges weakly to } x(t) \qquad \text{as } N \to \infty$$

where $x(t)$ is a Markov process on $S_\star$ with $x(0) = y$ and transition rates

$$p(x,y) = r(x,y)$$

Example



Proof strategy

If $r(\cdot, \cdot)$ is symmetric ($S = S_\star$), cite Grosskinsky, Redig, Vafayi, 2013
They analyze directly rescaled generator

Otherwise, martingale approach Beltrán, Landim, 2010
Potential theory combined with martingale arguments

Successfully applied to zero range process Beltrán, Landim, 2012

Capacities in inclusion process

Transition rates follow from capacities between subsets of metastable states

Proposition

Let $S_\star^1 \subsetneq S_\star$ *and* $S_\star^2 = S_\star \setminus S_\star^1$*. Then, for* $d_N \log N \to 0$,

$$\lim_{N\to\infty} \frac{1}{d_N} \mathrm{Cap}_N\Big(\bigcup_{x\in S_\star^1} \{\eta^{x,N}\}, \bigcup_{y\in S_\star^2} \{\eta^{y,N}\}\Big) = \frac{1}{|S_\star|} \sum_{x\in S_\star^1} \sum_{y\in S_\star^2} r(x,y)$$

From this one can compute

$$\lim_{N\to\infty} \frac{1}{d_N} p_N(\eta^{x,N}, \eta^{y,N}) \to r(x,y)$$

Capacity estimates using Dirichlet principle

Capacities satisfy

$$\mathrm{Cap}_N(A,B) = \inf\{D_N(F) : F(\eta) = 1\ \forall\, \eta \in A, F(\eta) = 0\ \forall\, \eta \in B\}$$

where $D_N(F)$ is the Dirichlet form

$$D_N(F) = \frac{1}{2}\sum_{\eta} \mu_N(\eta) \sum_{x,y\in S} \eta_x(d_N + \eta_y) r(x,y)[F(\eta^{x,y}) - F(\eta)]^2$$

Intuition for capacity estimates

All N particles are on at most 2 neighboring vertices in $S_\star$

After first particle escapes condensate, we have a 1D symmetric random walk

$$\eta_x(\eta_y + d_N)r(x,y) \approx \eta_y(\eta_x + d_N)r(y,x), \qquad \text{for } x,y \in S_\star$$

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For lower bound:

- Remove all other states and particle jumps from $D_N(F)$
- $D_N(F)$ reduces to weighted sum of Dirichlet forms of 1D random walks
- These can be explicitly minimized

Upper bound on Dirichlet form

Need to construct test function $F(\eta)$

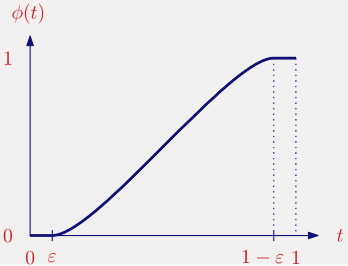
Good guess inside tubes $\eta_x + \eta_y = N$: $F(\eta) \approx \eta_x / N$

In fact better to choose smooth monotone function $\phi(t), t \in [0,1]$ with

$\phi(t) = 1 - \phi(1-t)\ \forall t \in [0,1]$
$\phi(t) = 0$ if $t \leq \varepsilon$

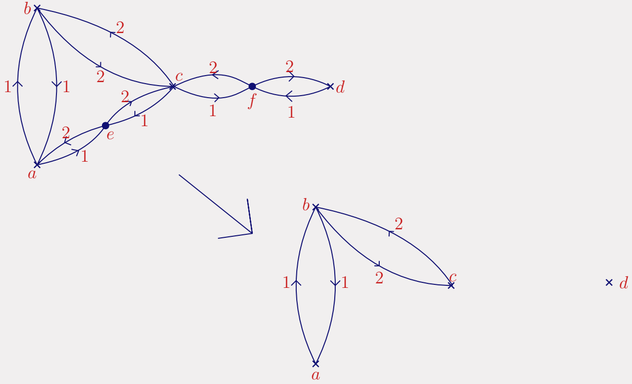
and set

$$F(\eta) = \sum_{x \in S_\star^1} \phi\left(\eta_x / N\right)$$



Longer time scales

If induced random walk on $S_\star$ is not connected, condensate jumps on longer time scales



Longer time scales

If induced random walk on $S_\star$ is not connected, condensate jumps on longer time scales

Consider underlying random walk on line with

$$S = \{1, \dots, L\}, \quad S_\star = \{1, L\}, \quad r(x,y) \neq 0 \Leftrightarrow |x-y| = 1, \quad m(2) = \dots = m(L-1)$$

Theorem (Bianchi, D., Giardinà, 2017)

For $L = 3$, condensate jumps at time scale N/d_N^2

For $L > 3$, condensate jumps at time scale N^2/d_N^3

Conjecture: these are only two extra time scales in reversible inclusion process

Symmetric IP with reservoirs

Again, consider random walk on line of length L

Symmetric random walk with for simplicity $r(x, x+1) = r(x+1, x) = 1$

Attach reservoirs to sites 1 and L with densities ρ_a and ρ_b, respectively

Generator:

$$\mathcal{L}f(\eta) = \sum_{x=1}^{L-1} \Big\{ \eta_x (d_N + \eta_{x+1}) \big[f(\eta^{x,x+1}) - f(\eta) \big] + \eta_{x+1} (d_N + \eta_x) \big[f(\eta^{x+1,x}) - f(\eta) \big] \Big\}$$
$$+ \rho_a (d_N + \eta_1) \big[f(\eta^{1,+}) - f(\eta) \big] + \eta_1 (d_N + \rho_a) \big[f(\eta^{1,-}) - f(\eta) \big]$$
$$+ \rho_b (d_N + \eta_L) \big[f(\eta^{L,+}) - f(\eta) \big] + \eta_L (d_N + \rho_b) \big[f(\eta^{L,-}) - f(\eta) \big]$$

Known results

Moments can be computed using duality Carinci, Giardinà, Giberti, Redig, 2013

$$\langle \eta_x \rangle = \rho_a + \frac{x}{L+1}(\rho_a - \rho_b)$$

If $\rho_a > \rho_b$ one expects a flow of particles from a to b

System is non-reversible, so difficult to analyze

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If $\rho_a = \rho_b$ we have an explicit reversible product measure

If $d \log(1/d)\, L \xrightarrow{L\to\infty} 0$, system is empty w.h.p. Grosskinsky, Redig, Vafayi, 2011

Least likely other configuration: one occupied site $\longrightarrow$ 'condensation'

Metastability

Metastable state if a condensate is on $2, \ldots, L-1$

Condensate performs random walk on $2, \ldots, L-1$ at rate $1/d_L$ (independent of size of condensate)

On time scale $1/d_L$, condensate on $3, \ldots, L-2$ does not change size

If only particles on 1 or L we have a fast process until site is empty

Formation of condensate

How long does it take to go from empty system to condensate on site 2 or $L-1$.

Heuristics: this happens on time scale $\frac{1}{d^2 \log(1/d)}$

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Open problems:

- Prove this (2D random walk in first quadrant)
- What is process at which condensate on 2 changes size?
- Couple together ingredients to study $\rho_a > \rho_b$
- Study current in this situation: large condensate occasionally traverses system (?)