Metastability on random graphs

Frank den Hollander, Leiden University, The Netherlands

Universiteit Leiden

Analysis and Simulations of Metastable Systems,
CIRM, Marseille, France, 3–7 April 2023.

§ WHAT IS METASTABILITY?

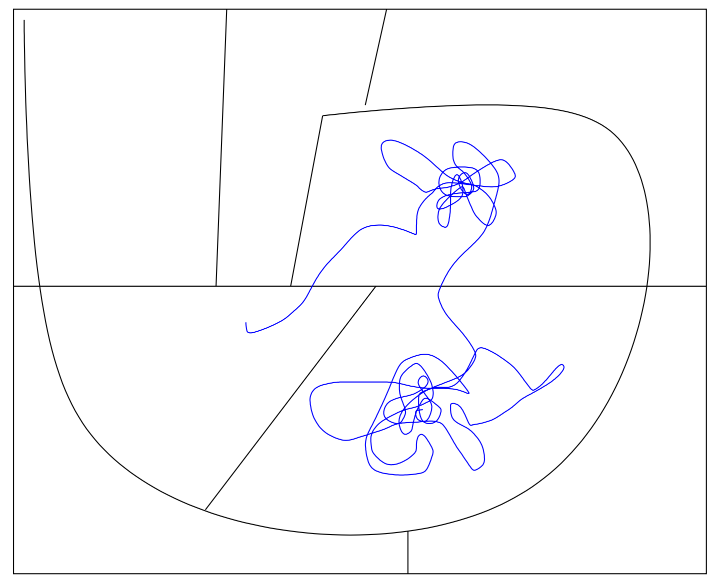
Metastability is the phenomenon where a

physical/chemical/biological system

under the influence of a stochastic dynamics makes slow transitions between different phases.

> The challenge is to propose mathematical models and to explain the experimentally observed universality.

Within the narrower perspective of statistical physics, the phenomenon of metastability is a dynamical manifestation of a first-order phase transition. Well-known examples are condensation and magnetisation.



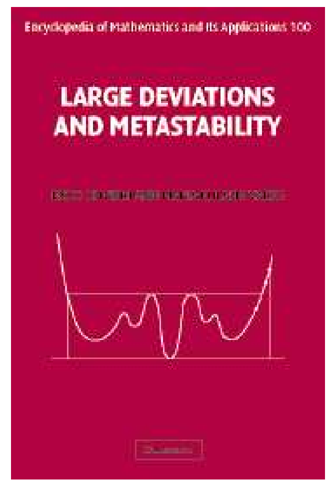
Fast transitions within phases.
Slow transitions between phases.

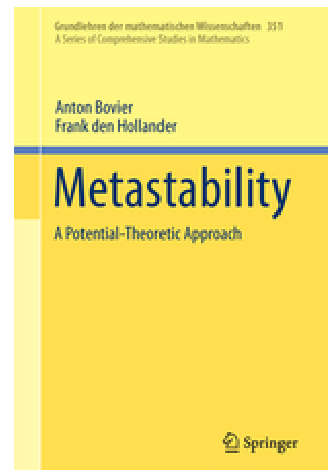
There is a vast literature describing concrete examples. Much of the literature is summarised in two monographs, which focus on two different approaches to metastability, based on large deviation theory and potential theory.

MONOGRAPHS:

Olivieri, Vares 2005

Bovier, den Hollander 2015





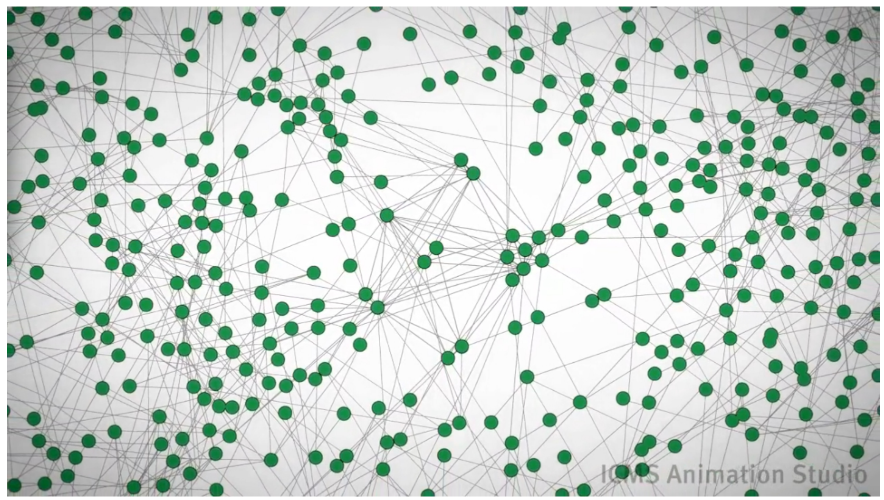
Since then the field has moved on is developing rapidly.

Other approaches have been successful as well:

spectral approach
martingale approach
computational approach
...

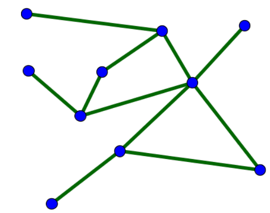
All of the approaches will appear in talks in this workshop.

A recent development is the study of metastability on random graphs, which is a natural spin-off in the area of complex networks.



In this talk we focus on a particular class of examples.

§ GLAUBER DYNAMICS ON GRAPHS



Let $G = (V, E)$ be a connected graph. Ising spins are attached to the vertices V and interact with each other along the edges E.

1. The energy associated with the configuration $\sigma = (\sigma_i)_{i \in V} \in \Omega = \{-1, +1\}^V$ is given by the Hamiltonian

$$H(\sigma) = -J \sum_{(i,j) \in E} \sigma_i \sigma_j - h \sum_{i \in V} \sigma_i$$

where $J > 0$ is the ferromagnetic interaction strength and $h > 0$ is the external magnetic field.

2. Spins flip according to Glauber dynamics

$$\forall \sigma \in \Omega \ \forall j \in V: \quad \sigma \to \sigma^j \text{ at rate } e^{-\beta[H(\sigma^j)-H(\sigma)]_+}$$

where σ^j is the configuration obtained from σ by flipping the spin at vertex j, and $\beta > 0$ is the inverse temperature.

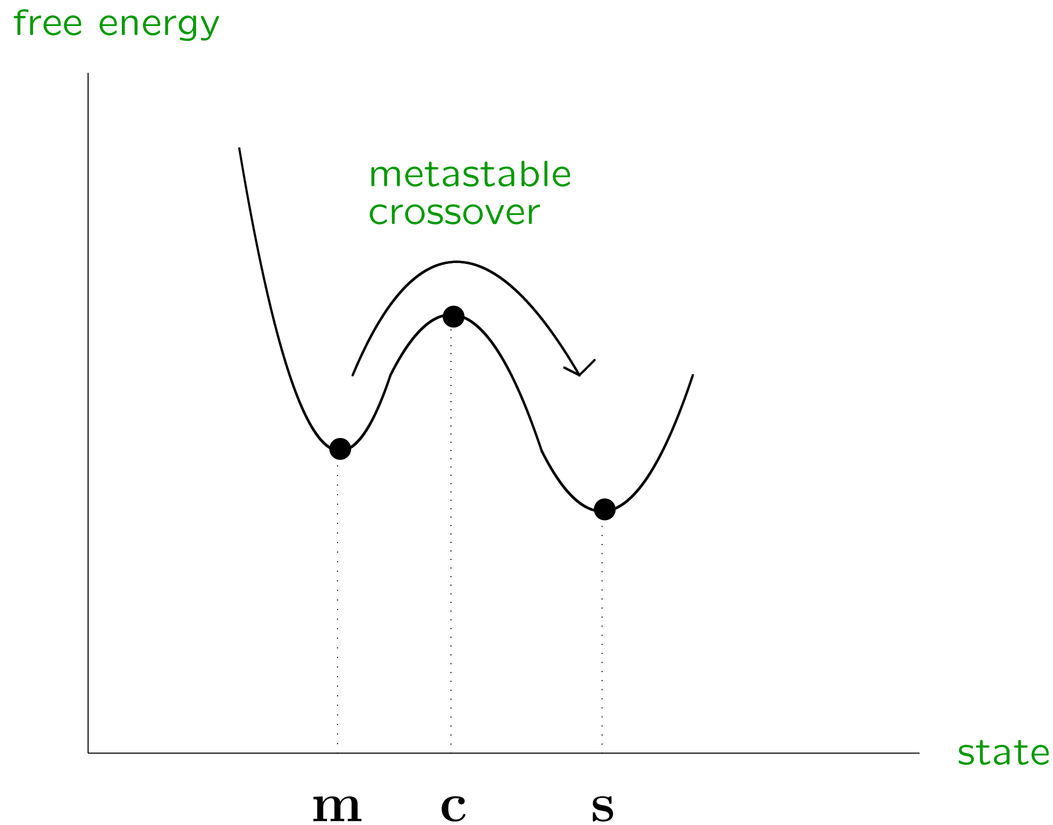
3. The Gibbs measure

$$\mu(\sigma) = \frac{1}{\Xi} e^{-\beta H(\sigma)}, \qquad \sigma \in \Omega,$$

is the reversible equilibrium of this dynamics.

4. Three sets of configurations play a central role:

$$\mathbf{m} = \text{metastable state}$$
$$\mathbf{c} = \text{crossover state}$$
$$\mathbf{s} = \text{stable state}.$$



Caricature of the free energy landscape

– energy and entropy –

The goal of this talk is to investigate what can be said when G is a random graph.

This target is to derive an Arrhenius law of the form

$$\mathbb{E}_{\mathbf{m}}[\tau_{\mathrm{s}}] = K\,\mathrm{e}^{N\Gamma}[1 + o(1)], \qquad N \to \infty.$$

In general,

$$\Gamma = \Gamma(J, h, \beta), \qquad K = K(J, h, \beta),$$

are random and are generally hard to identify. In fact, most results to date are bounds on these quantities.

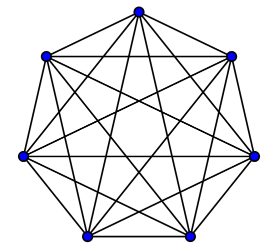
§ EXAMPLES OF RANDOM GRAPHS

- Erdős-Rényi random graphs
- Chung-Lu random graphs
- Galton-Watson random trees
- Configuration random graphs

§ COLLABORATIONS

Anton Bovier
Sander Dommers
Frank den Hollander
Oliver Jovanovski
Saeda Marello
Francesca Nardi
Elena Pulvirenti
Martin Slowik
Siamak Taati
...

§ QUENCHED VERSUS ANNEALED



Take the complete graph with N vertices. Note that this graph is dense. Consider the Hamiltonians

$$H_N^{\text{que}}(\sigma) = -\frac{1}{2N} \sum_{1\le i<j\le N} J_{i,j}\sigma_i\sigma_j - h \sum_{1\le i\le N} \sigma_i,$$

$$H_N^{\text{ann}}(\sigma) = -\frac{1}{2N} \sum_{1\le i<j\le N} \mathbb{E}_J[J_{i,j}]\sigma_i\sigma_j - h \sum_{1\le i\le N} \sigma_i,$$

where $(J_{ij})_{1\le i<j\le N}$ are i.i.d. random variables with joint law $\mathbb{P}_J$.

The annealed Hamiltonian corresponds to the Curie-Weiss model, which is well understood.

The metastable regime corresponds to the region of the parameters for which $(\mathbf{m}, \mathbf{s})$ forms a metastable pair, i.e., transitions towards $\{\mathbf{m}, \mathbf{s}\}$ are much faster than transition between $\mathbf{m}$ and $\mathbf{s}$.

For the model under consideration the metastable regime is

$$\beta > \beta_c(\mathbb{P}_J), \qquad 0 \leq h < h_c(\beta, \mathbb{P}_J), \qquad N \to \infty,$$

THEOREM Bovier, den Hollander, Marello, Pulvirenti, Slowik 2022

If the annealed system is metastable, then the quenched system is metastable with $\mathbb{P}_J$*-probability tending to* 1.

THEOREM Bovier, den Hollander, Marello, Pulvirenti, Slowik 2022

Under certain technical conditions, for all $u > 0$,

$$\lim_{N\to\infty} \mathbb{P}_J \left(C_1 e^{-u} \leq \frac{\mathbb{E}^{\text{que}}_{\mathbf{m}_N}(\tau_{\mathbf{s}_N})}{\mathbb{E}^{\text{ann}}_{\mathbf{m}_N}(\tau_{\mathbf{s}_N})} \leq C_2 e^{u} \right) \geq 1 - k_1 e^{-k_2 u^2},$$

with

- $\mathbf{m}_N, \mathbf{s}_N$ *metastable and stable states for finite* N,
- $k_1, k_2 > 0$ *absolute constants,*
- C_1, C_2 *constants that depend on* $\beta, h, \mathbb{P}_J$.

Details are explained in the talk by Elena Pulvirenti!

§ TECHNIQUES

Proofs rely on elaborate techniques:

isoperimetric inequalities
concentration estimates
capacity estimates
coupling techniques
coarse-graining techniques
. . .

These techniques exploit the fact that in the dense regime the random graph is locally homogeneous.

§ OPEN QUESTIONS

- Can the prefactors of the annealed and the quenched system be computed?
- What about partial annealing?
  [Talk Elena Pulvirenti]
- What about non-dense graphs?

TAKE-HOME MESSAGE

Prefactors of average metastable crossover times are delicate objects for random graphs, because they depend in an intricate manner on the underlying geometry.

Very little is known so far and much remains to be done!