Real diffusion with complex spectral gap

Jean-François Bony and Laurent Michel

I

Setting

Generator of Langevin processes

- Typically, the generator of a Langevin process writes

$$P = -h\nabla a h\nabla + \frac{1}{2}\left(b \cdot h\nabla + h\nabla \cdot b\right) + c$$

with real coefficients $a(x, h), b(x, h), c(x, h)$ and a symmetric. The parameter h is proportional to the temperature, and we will work in the low temperature (or semiclassical) regime $h \to 0$. In general,

- an (hypo-)elliptic assumption guarantees that P is maximal accretive
- a confining assumption guarantees that the spectrum P has only eigenvalues near 0
- the process has an invariant Gibbs measure, that is there exists a function f such that

$$Pe^{-f/h} = P^* e^{-f/h} = 0$$

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- When $b \neq 0$, the operator P is not selfadjoint.

Does its implies that P has non-real spectrum ?

Non-selfadjoint operators can have real spectrum

- Consider the reversible overdamped Langevin diffusion on a compact manifold M

$$dX_t = -2\nabla f(X_t) + \sqrt{2h}\, dW_t$$

Its generator is the Kramers–Smoluchovski operator

$$\mathcal{L} = -2\nabla f \cdot \nabla + h\Delta$$

which is non-selfadjoint on $H^2(M)$.

- But writing $P = -he^{-f/h}\mathcal{L}e^{f/h}$, we get the Witten Laplacian

$$P = -h^2\Delta + |\nabla f|^2 - h\Delta f$$

which is selfadjoint on $H^2(M)$. Since $\mathcal{L}$ and P have the same spectrum,

the spectrum of $\mathcal{L}$ is real.

- Note that

$$Pe^{-f/h} = P^* e^{-f/h} = 0$$

Consequences of the Eyring–Kramers law

- $P = -h\nabla a h\nabla + \frac{1}{2}\big(b\cdot h\nabla + h\nabla\cdot b\big) + c$ and $Pe^{-f/h} = P^{*}e^{-f/h} = 0$.
- We assume that f is a Morse function ($\nabla f(x) = 0 \implies \operatorname{Hess} f(x)$ is invertible). Let $1 \leq n_0 < +\infty$ be the number of (local) minima of f.

Theorem (the Eyring–Kramers law)

For h small enough, the spectrum of P satisfies

1) P has n_0 exponentially small eigenvalues $\lambda_1, \dots, \lambda_{n_0}$,

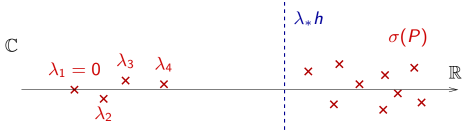
2) $\lambda_1 = 0$ and $\operatorname{Ker} P_0 = e^{-f/h}\mathbb{C}$ (Perron–Frobenius theorem),

3) the rest of the spectrum of P lies in $\{\operatorname{Re} z > \lambda_* h\}$ with $\lambda_* > 0$,

4) for all $1 \leq n \leq n_0$, we have

$$\lambda_n \approx \sum_{k=0}^{+\infty} a_k^n h^{k+1} e^{-2S_n/h}$$

for some $S_n = f(s_n) - f(m_n)$ and $a_k^n \in \mathbb{R}$.



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- Proved under various assumptions in the present setting by Bovier, Eckhoff, Gayrard and Klein ('04), Helffer, Klein and Nier ('04), Bovier, Gayrard and Klein ('05), Hérau, Hitrik and Sjöstrand ('11), Berglund ('13), Bouchet and Reygner ('16), Landim, Mariani and Seo ('19), Le Peutrec and Michel ('20), Nectoux ('21), Lee and Seo ('21), Normand ('23), ...

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- We always have $\lambda_1 = 0 \in \mathbb{R}$ and

$$|\operatorname{Im} \lambda_n| = \mathcal{O}(h^\infty) \operatorname{Re} \lambda_n$$

$\implies$ one can never deduce from the Eyring–Kramers law that λ_n is non-real

$\implies$ if $\lambda_n \notin \mathbb{R}$, its imaginary part is extremely small

Consequences of the complexe conjugation

- Since $P = -h\nabla a h\nabla + \frac{1}{2}(b \cdot h\nabla + h\nabla \cdot b) + c$ with a, b, c real valued,

$$\overline{Pu} = P\overline{u}$$

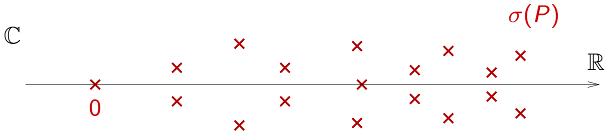
for all $u \in \mathcal{D}(P)$. In particular,

$$Pu = \lambda u \quad \Longrightarrow \quad P\overline{u} = \overline{\lambda}\overline{u}$$

It implies

Principle

$$\lambda \in \sigma(P) \quad \Longleftrightarrow \quad \overline{\lambda} \in \sigma(P)$$



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$\Longrightarrow$ if λ_n is the unique eigenvalue with the expansion $\sum a_k^n h^{k+1} e^{-2S_n/h}$, then λ_n is real

$\Longrightarrow$ if all the S_n's are different, then all the λ_n are real

$\Longrightarrow$ if $n_0 = 2$, then λ_2 is real (and $\lambda_1 = 0 \in \mathbb{R}$)

Is it possible to have non-real spectrum ?

II

Our reference operator

The Morse function f

On $\mathbb{R}^2$, let f be a smooth Morse function with $f(x) = x^2$ at infinity and invariant under R where

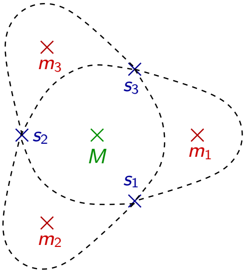
$$R \text{ is the rotation of angle } 2\pi/3 \text{ around } 0$$

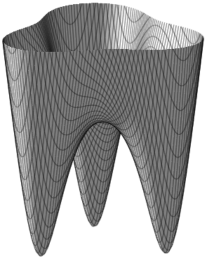
The critical points of f are

3 (global) minima m_1, m_2, m_3

3 saddle points s_1, s_2, s_3

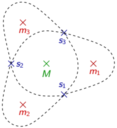
1 (local) maximum $M = 0$

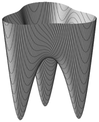




The operator P_0

- The function f:





- As reference (or unperturbed) operator, we choose the Witten Laplacian

$$P_0 = d_f^* \circ d_f = -h^2\Delta + |\nabla f|^2 - h\Delta f$$

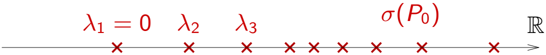
with

$$d_f = e^{-f/h} \circ h\nabla \circ e^{f/h} = \begin{pmatrix} h\partial_{x_1} + \partial_{x_1} f \\ h\partial_{x_2} + \partial_{x_2} f \end{pmatrix}$$

- P_0 is self-adjoint with domain $\mathscr{D}(P_0) = H^2(\mathbb{R}^2) \cap \langle x \rangle^{-2} L^2(\mathbb{R}^2)$, has a compact resolvent, $P_0 \geq 0$ and

$$\operatorname{Ker} P_0 = e^{-f/h}\mathbb{C}$$

$\Longrightarrow$ The spectrum of P_0 is



purely real without Jordan's block.

The spectrum of P_0

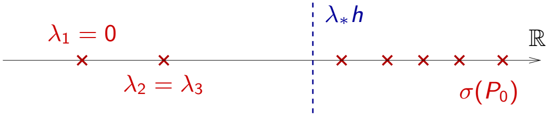
- $P_0 = d_f^* \circ d_f = -h^2\Delta + |\nabla f|^2 - h\Delta f$

Proposition (low-lying eigenvalues of P_0)

For h small enough, P_0 has 3 exponentially small eigenvalues $\lambda_1(h), \lambda_2(h), \lambda_3(h)$ and the rest of its spectrum is in $[\lambda_* h, +\infty[$ for some $\lambda_* > 0$.

$$\lambda_1 = 0 \qquad \lambda_2 = \lambda_3 \qquad \textit{and} \qquad \lambda_2 \sim \frac{3|\mu(s)||\operatorname{Hess} f(m)|^{1/2}}{\pi|\det \operatorname{Hess} f(s)|^{1/2}} h e^{-2S/h}$$

In the Eyring–Kramers formula, $S = f(s) - f(m) > 0$ and $\mu(s) < 0$ denotes the unique negative eigenvalue of $\operatorname{Hess} f(s)$.



- Mainly proved by Hérau, Hitrik and Sjöstrand ('11) and Michel ('19), the unique novelty is that λ_2 has multiplicity two.

"Proof" that $\lambda_2 = \lambda_3$

- By contradiction: assume that λ_2 has multiplicity 1 and let $u \in \mathrm{Ker}(P_0 - \lambda_2)$.
- Consider the 3 functions

$$\varphi_1 = \chi_1 e^{-f/h} \qquad \varphi_2 = \chi_2 e^{-f/h} \qquad \varphi_3 = \chi_3 e^{-f/h}$$

where χ_1 localizes near m_1, $\chi_2 = \chi_1 \circ R$ and $\chi_3 = \chi_1 \circ R^2$. We work as if

$$(\varphi_1, \varphi_2, \varphi_3) \text{ is a basis of the eigenspace associated to } \{\lambda_1, \lambda_2, \lambda_3\}$$

- Then, there exist $u_1, u_2, u_3 \in \mathbb{R}$ such that

$$u = u_1\varphi_1 + u_2\varphi_2 + u_3\varphi_3$$
$$u \circ R = u_3\varphi_1 + u_1\varphi_2 + u_2\varphi_3$$

- Since P_0 is stable by the rotation R, $u \circ R$ is also an eigenvector associated to the simple eigenvalue λ_2. Thus, $u \circ R = \alpha u$ for some $\alpha \in \mathbb{R}$. That is

$$u_3 = \alpha u_1 \qquad u_1 = \alpha u_2 \qquad u_2 = \alpha u_3$$

then $\alpha^3 = 1$ and eventually $\alpha = 1$. Thus, $u = u_1(\varphi_1 + \varphi_2 + \varphi_3)$.

- But, u must be orthogonal to $e^{-f/h} \approx \varphi_1 + \varphi_2 + \varphi_3$.

$\Longrightarrow$ Contradiction.

III

Operators with non-real eigenvalues

We perturb P_0 by an anti-adjoint operator of order 1. More precisely, consider

$$P_{\mathrm{com}} = P_0 + B \qquad \text{with} \qquad B = \frac{1}{2}\big(b \cdot h\nabla + h\nabla \cdot b\big)$$

for some $b(x,h) \in C_0^\infty(\mathbb{R}^2;\mathbb{R}^2)$ such that $Be^{-f/h} = 0$. Then,

$$P_{\mathrm{com}}e^{-f/h} = P_{\mathrm{com}}^* e^{-f/h} = 0$$

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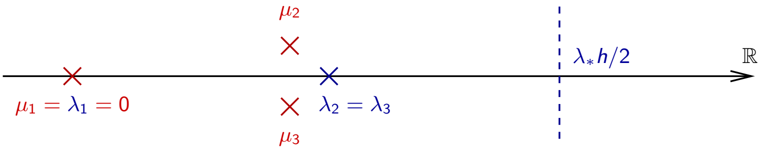
Theorem (Non-real eigenvalues)

Let $r(h) = \mathcal{O}(h^\infty)$ be positive. For h small enough, there exists $b(x,h) \in S(r)$ as before such that

$$\sigma(P_{\mathrm{com}}) \cap \{z \in \mathbb{C};\ \operatorname{Re} z < \lambda_* h/2\} = \{\mu_1(h), \mu_2(h), \mu_3(h)\}$$

with $\mu_1 = 0$, $\mu_2 = \lambda_2 + \mathcal{O}(r)$, $\mu_3 = \overline{\mu_2}$ and

$$\operatorname{Im} \mu_2 \neq 0$$



Time evolution

- The time evolution equation associated to P_{com} is the Fokker–Planck equation

$$(\mathrm{FP}) \qquad \begin{cases} h\partial_t u(t,x) = -P_{\mathrm{com}} u(t,x) \\ u(0,x) = u_0(x) \end{cases}$$

where $u_0(x) \in L^2(\mathbb{R}^2;\mathbb{R})$ is the initial data.

Corollary (Metastability)

The solution $u = e^{-tP_{\mathrm{com}}/h} u_0$ of (FP) can be written

$$\begin{aligned} e^{-tP_{\mathrm{com}}/h} u_0 &= u_1 + e^{-t\mu_2/h} u_2 + e^{-t\mu_3/h} u_3 + \varepsilon(t) \\ &= u_1 + e^{-t\,\mathrm{Re}\,\mu_2/h}\big(\cos(t\,\mathrm{Im}\,\mu_2/h)u_{\mathrm{c}} + \sin(t\,\mathrm{Im}\,\mu_2/h)u_{\mathrm{s}}\big) + \varepsilon(t) \end{aligned}$$

where u_j, $j = 1,2,3$, is the spectral projection of u_0 on the eigenspace of P_{com} associated to μ_j, $u_{\mathrm{c}} = u_2 + u_3$, $u_{\mathrm{s}} = iu_3 - iu_2$ and

$$\|\varepsilon(t)\|_{L^2(\mathbb{R}^2)} \leq Ce^{-t/C}\|u_0\|_{L^2(\mathbb{R}^2)}$$

for some constant $C > 0$ independent of t, h, u_0.

- $(u_{\mathrm{c}}, u_{\mathrm{s}})$ are real-valued and form a basis of $\mathrm{Im}\,\mathbb{1}_{\mu_2}(P_{\mathrm{com}}) \oplus \mathrm{Im}\,\mathbb{1}_{\mu_3}(P_{\mathrm{com}})$ iff u_2, u_3, u_{c} or u_{s} does not vanish identically. Also $u_3 = \overline{u_2}$.

Sketch of proof

- We consider the operator

$$P_\varepsilon = P_0 + B$$

with the perturbation

$$B = \varepsilon \mathscr{B} \qquad \text{and} \qquad \mathscr{B} = d_f^* \circ \begin{pmatrix} 0 & g \\ -g & 0 \end{pmatrix} \circ d_f$$

for some constant $\varepsilon(h) > 0$ and some function $g(x,h) \in C_0^\infty(\mathbb{R}^2;\mathbb{R})$.

- Direct computations give $\mathscr{B} e^{-f/h} = 0$ and

$$\mathscr{B} = \frac{1}{2}\big(b \cdot h\nabla + h\nabla \cdot b\big) \qquad \text{with} \qquad b(x,h) = \begin{pmatrix} h\partial_{x_2} g - 2g\partial_{x_2} f \\ -h\partial_{x_1} g + 2g\partial_{x_1} f \end{pmatrix}$$

$\Longrightarrow$ B satisfies the required properties.

- We will obtain non-real eigenvalues using

the perturbation theory

More precisely, we show that, for all h fixed small enough, the operator P_ε satisfies the conclusions of the theorem for $\varepsilon > 0$ small enough.

- For h fixed small enough, P_0 has 3 small eigenvalues $\lambda_1 = 0, \lambda_2 = \lambda_3$. Let

(u, v) be an orthonormal basis of $\mathrm{Im}\, \mathbb{1}_{\lambda_2}(P_0)$ of real-valued functions

where $\mathbb{1}_{\lambda_2}(P_0)$ is the spectral projection of P_0 associated to λ_2.

$$\mathbb{1}_{\lambda_2}(P_0)\mathscr{B}\mathbb{1}_{\lambda_2}(P_0) \text{ in the basis } (u, v) = \begin{pmatrix} \langle \mathscr{B}u, u\rangle & \langle \mathscr{B}v, u\rangle \\ \langle \mathscr{B}u, v\rangle & \langle \mathscr{B}v, v\rangle \end{pmatrix} \in M_{2\times 2}(\mathbb{R})$$

Lemma

There exist $g(x, h) \in C_0^\infty(\mathbb{R}^2; \mathbb{R})$ and $\gamma(h) \in \mathbb{R} \setminus \{0\}$ such that

$$\mathbb{1}_{\lambda_2}(P_0)\mathscr{B}\mathbb{1}_{\lambda_2}(P_0) \text{ in the basis } (u, v) = \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix}$$

Proof: ▸ Since $\mathscr{B}$ is anti-adjoint, we have $\langle \mathscr{B}u, u\rangle = \langle \mathscr{B}v, v\rangle = 0$. It remains to choose $g(x, h)$ such that

$$\gamma = \langle \mathscr{B}v, u\rangle = \int_{\mathbb{R}^2} g e^{-2f/h}\left(\partial_{x_2} e^{f/h} v \partial_{x_1} e^{f/h} u - \partial_{x_1} e^{f/h} u \partial_{x_2} e^{f/h} v\right) dx \neq 0$$

This can not be done using the quasimodes constructed to prove the Eyring–Kramers law.

- We prove by contradiction that $\partial_{x_2} e^{f/h} v \partial_{x_1} e^{f/h} u - \partial_{x_1} e^{f/h} v \partial_{x_2} e^{f/h} u \not\equiv 0$. Thus, assume that

$$\partial_{x_2} V \partial_{x_1} U - \partial_{x_1} V \partial_{x_2} U \equiv 0 \qquad \text{with} \qquad U = e^{f/h} u, \;\; V = e^{f/h} v$$

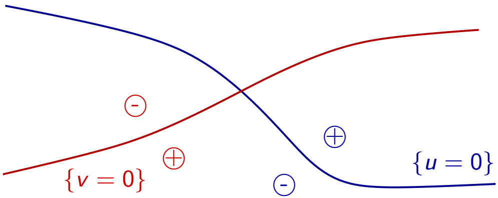
Geometrically, it means that U and V have the same level sets.

- As a particular level set, we use the nodal set

$$\{U = 0\} = \{u = 0\}$$

Theorem (Courant ('23), Cheng ('76))

1) *the nodal set $\{u = 0\}$ is a curve which separates $\mathbb{R}^2$ into 2 parts where u changes its sign (the same for v).*
2) *$\nabla u \neq 0$ on the nodal set $\{u = 0\}$ (the same for v).*
3) *the nodal sets of u and v cross, i.e. $\{u = 0\} \cap \{v = 0\} \neq \emptyset$.*



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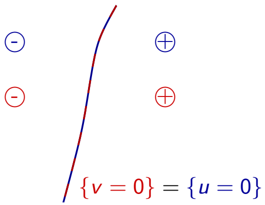

- Since U and V has the same level sets, they have the same nodal sets

$$\{u = 0\} = \{v = 0\}$$

- Modulo a change of sign, u and v have the same sign and then $\langle u, v \rangle \neq 0$.

$\Longrightarrow$ Contradiction.

- We have proved that

$$\partial_{x_2} e^{f/h} v \partial_{x_1} e^{f/h} u - \partial_{x_1} e^{f/h} v \partial_{x_2} e^{f/h} u \not\equiv 0$$

Since this function is continuous, there exists $g(x,h) \in C_0^\infty(\mathbb{R}^2;\mathbb{R})$ such that

$$\gamma = \langle \mathscr{B} v, u\rangle = \int_{\mathbb{R}^2} g e^{-2f/h} (\partial_{x_2} e^{f/h} v \partial_{x_1} e^{f/h} u - \partial_{x_1} e^{f/h} v \partial_{x_2} e^{f/h} u) dx \neq 0$$

and the lemma follows.

Lemma

$$\mathbb{1}_{\lambda_2}(P_0)\mathscr{B}\mathbb{1}_{\lambda_2}(P_0) \text{ in the basis } (u,v) = \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix} \text{ with } \gamma \neq 0$$

- The eigenvalues of $\mathbb{1}_{\lambda_2}(P_0)\mathscr{B}\mathbb{1}_{\lambda_2}(P_0)$ are $\pm i\gamma$. Then, by the perturbation theory, for ε small enough (depending on h), P_ε has 3 small eigenvalues

$$\begin{aligned} \mu_1(\varepsilon) &= 0 \\ \mu_2(\varepsilon) &= \lambda_2 + i\gamma\varepsilon + \mathcal{O}(\varepsilon^2) \\ \mu_3(\varepsilon) &= \lambda_2 - i\gamma\varepsilon + \mathcal{O}(\varepsilon^2) \end{aligned}$$

which are analytic in ε near 0.

- The theorem follows taking $P_{\mathrm{com}} = P_{\varepsilon(h)}$ with $\varepsilon(h) > 0$ small enough.

IV

Operators with a Jordan block

- We consider more general perturbations of P_0 of the form

$$P_{\mathrm{Jor}} = d_f^* \circ (1+\chi) Id \circ d_f + B \qquad \text{with} \qquad B = \frac{1}{2}\left(b \cdot h\nabla + h\nabla \cdot b\right)$$

for some $\chi(x,h) \in C_0^\infty(\mathbb{R}^2;\mathbb{R})$, $b(x,h) \in C_0^\infty(\mathbb{R}^2;\mathbb{R}^2)$ such that $Be^{-f/h} = 0$.

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Theorem (Jordan block)

Let $r(h) = \mathcal{O}(h^\infty)$ be positive. For h small enough, there exist $\chi, b \in S(r)$ as before such that

$$\sigma(P_{\mathrm{Jor}}) \cap \{z \in \mathbb{C};\ \mathrm{Re}\, z < \lambda_* h/2\} = \{\lambda_1 = 0, \lambda_2 = \lambda_3\}$$

and P_{Jor} has a non-zero Jordan block associated with λ_2.

- There exists a real-valued orthonormal basis (e_1, e_2) of $\mathrm{Im}\, \mathbb{1}_{\lambda_2}(P_{\mathrm{Jor}})$ s. t.

$$\mathbb{1}_{\lambda_2}(P_{\mathrm{Jor}}) P_{\mathrm{Jor}} \mathbb{1}_{\lambda_2}(P_{\mathrm{Jor}}) \text{ in the basis } (e_1, e_2) = \begin{pmatrix} \lambda_2 & \rho \\ 0 & \lambda_2 \end{pmatrix} \text{ with } \rho \in \mathbb{R} \setminus \{0\}$$

- As for the non-real eigenvalues, we have

$$|\rho| = \mathcal{O}(h^\infty)\lambda_2$$

Return to equilibrium

- The time evolution equation associated to P_{Jor} is the Fokker–Planck equation

$$(\mathrm{FP}) \qquad \begin{cases} h\partial_t u(t,x) = -P_{\mathrm{Jor}} u(t,x) \\ u(0,x) = u_0(x) \end{cases}$$

where $u_0(x) \in L^2(\mathbb{R}^2;\mathbb{R})$ is the initial data.

Corollary (Metastability)

The solution $u = e^{-tP_{\mathrm{com}}/h} u_0$ of (FP) can be written

$$e^{-tP_{\mathrm{com}}/h} u_0 = u_1 + t\, e^{-t\lambda_2/h} u_2 + e^{-t\lambda_2/h} u_3 + \varepsilon(t)$$

where $u_1 = \mathbb{1}_{\lambda_1}(P_{\mathrm{com}}) u_0$, $u_2 = \rho\langle e_2, \mathbb{1}_{\lambda_2}(P_{\mathrm{com}}) u_0\rangle e_1$, $u_3 = \mathbb{1}_{\lambda_2}(P_{\mathrm{com}}) u_0$ and

$$\|\varepsilon(t)\|_{L^2(\mathbb{R}^2)} \leq C e^{-t/C} \|u_0\|_{L^2(\mathbb{R}^2)}$$

for some constant $C > 0$ independent of t, h, u_0.

- As a consequence, the sharp return to equilibrium is

$$\left\| e^{-tP_{\mathrm{Jor}}/h} - \mathbb{1}_{\lambda_1}(P_{\mathrm{com}}) \right\| \sim \nu\, t\, e^{-t\lambda_2/h} \qquad \text{as } t \to +\infty$$

for some $\nu(h) > 0$.

Sketch of proof

- It is more difficult to have a Jordan block than non-real eigenvalues. Consider

$$Q(\varepsilon) = \begin{pmatrix} \lambda_2 & \varepsilon \\ 0 & \lambda_2 + \varepsilon^2 \end{pmatrix}$$

which is analytic in ε. At the first order

$$Q(\varepsilon) \approx \begin{pmatrix} \lambda_2 & \varepsilon \\ 0 & \lambda_2 \end{pmatrix}$$

has a Jordan block for $\varepsilon \neq 0$, but this is never the case for $Q(\varepsilon)$.

- For $\tau = (\tau_1, \tau_2, \tau_3, \tau_4) \in \mathbb{R}^4$ small, we consider

$$P(\tau) = d_f^* \circ (1 + \tau_1 \chi_1 + \tau_2 \chi_2 + \tau_3 \chi_3) Id \circ d_f + \tau_4 \mathscr{B}$$

where $\chi_j \in C_0^\infty(\mathbb{R}^2; [0,1])$ is localized near the saddle point s_j and $\mathscr{B}$ is as before.

- As before, we use the perturbation theory with respect to τ at each h fixed small enough.

- For τ small enough, $P(\tau)$ has 2 eigenvalues $\lambda_2(\tau), \lambda_3(\tau)$ near λ_2 and

$$\lambda_j(\tau) \xrightarrow{\tau \to 0} \lambda_2 \qquad \text{for } j = 2, 3$$

Let $Q(\tau)$ be the matrix of $P(\tau)$ restricted to its eigenspace associated to $\{\lambda_2(\tau), \lambda_3(\tau)\}$ and expressed in an appropriate basis. Then

$$\mathbb{R}^4 \ni \tau \longmapsto Q(\tau) \in M_{2\times 2}(\mathbb{R})$$

is analytic and $Q(0) = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Denote $C = \dfrac{|\mu(s)||\,\text{Hess}\, f(m)|^{1/2}}{4\pi|\det \text{Hess}\, f(s)|^{1/2}} h e^{-2S/h}$

Lemma

$$\partial_{\tau_1} Q(0) = C \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \qquad \partial_{\tau_2} Q(0) = C \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix}$$

$$\partial_{\tau_3} Q(0) = C \begin{pmatrix} 3 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \qquad \partial_{\tau_4} Q(0) = \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix}$$

Its proof uses the sharp quasimodes of Le Peutrec and Michel ('20) for P_0.

- Note that $(\partial_{\tau_j} Q(0))_{j=1,2,3,4}$ form a basis of $M_{2\times 2}(\mathbb{R})$.

- In other words,

$$d_0Q : \mathbb{R}^4 \simeq T_0\mathbb{R}^4 \longrightarrow T_{\lambda_2 Id}M_{2\times 2}(\mathbb{R}) \simeq M_{2\times 2}(\mathbb{R})$$

is an isomorphism. By the inverse function theorem,

$$\mathbb{R}^4 \ni \tau \longmapsto Q(\tau) \in M_{2\times 2}(\mathbb{R})$$

is a local diffeomorphism from a neighborhood of 0 to a neighborhood of $\lambda_2 Id$.

- Then, there exists $\tau(h) \in \mathbb{R}^4$ with $|\tau| < r$ such that

$$Q(\tau) = \begin{pmatrix} \lambda_2 & \rho \\ 0 & \lambda_2 \end{pmatrix}$$

for some $\rho(h) \neq 0$. Taking $P_{\mathrm{Jor}} = P(\tau)$, the theorem follows since $Q(\tau)$ is P_{Jor} restricted to its stable eigenspace $\mathrm{Im}\, \mathbb{1}_{\lambda_2}(P_{\mathrm{Jor}})$.

- Similar ideas have been by Sjöstrand ('87) for resonances.