Sharp convergence in the total variation distance for Langevin dynamics

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Plan

Mixing-time

Mixing-time

Well-posed problem

1. Let $(X_t(x_0))_{t\geq 0}$ be a dynamical system (model) such that the current state at time t, $X_t(x_0)$, converges to its limiting state μ as $t \to \infty$ for some suitable (distance/discrepancy) dist.
2. Assume that the map $t \mapsto \mathrm{dist}(X_t(x_0), \mu)$ is non-increasing.
3. Well-posed problem: given a prescribed error $\eta > 0$, the η-mixing-time is defined as

$$\tau_\eta(x_0) := \inf\{t \geq 0 : \mathrm{dist}(X_t(x_0), \mu) \leq \eta\}.$$

It is a hard problem in general! For instance, in ergodic Markov chains theory we have the existence of constants $C(x_0, N)$ and $\delta(N)$ satisfying

$$\left\| X_t^N(x_0) - \mu^N \right\|_{\mathbb{TV}} \leq C(x_0, N) e^{-\delta(N) t} \quad \text{for all} \quad t \geq 0.$$

Here N denotes the cardinality of the space state. Good knowledge of $C(x_0, N)$ and $\delta(N)$ is needed. What about lower bound estimates?

A warm-up example: the Ornstein–Uhlenbeck process

Let $(X_t(x_0))_{t\geq 0}$ be the solution of the SDE

$$\mathrm{d}X_t = -\lambda X_t \mathrm{d}t + \sqrt{\epsilon}\mathrm{d}W_t, \qquad X_0 = x_0,$$

with λ and ϵ being positive numbers. Integration by parts formula yields

$$X_t(x_0) = e^{-\lambda t}x_0 + \sqrt{\epsilon}e^{-\lambda t}\int_0^t e^{\lambda s}\mathrm{d}W_s.$$

Then Itô's isometry implies

$$X_t(x_0) \stackrel{\mathcal{D}}{=} \mathcal{N}\left(e^{-\lambda t}x_0, \frac{\epsilon}{2\lambda}(1-e^{-2\lambda t})\right),$$

and the limiting law is given by

$$\mu \stackrel{\mathcal{D}}{=} \mathcal{N}\left(0, \frac{\epsilon}{2\lambda}\right).$$

OUP total variation mixing time

For computing the η-mixing time, one needs to find the smallest $t > 0$ for which

$$\|X_t(x_0) - \mu\|_{\mathbb{TV}} = \left\| \mathcal{N}\left(e^{-\lambda t}x_0, \frac{\epsilon}{2\lambda}(1-e^{-2\lambda t})\right) - \mathcal{N}\left(0, \frac{\epsilon}{2\lambda}\right) \right\|_{\mathbb{TV}}$$
$$= \left\| \mathcal{N}\left(\sqrt{\frac{2\lambda}{\epsilon}}e^{-\lambda t}x_0, 1-e^{-2\lambda t}\right) - \mathcal{N}(0,1) \right\|_{\mathbb{TV}} \leq \eta.$$

Since

$$|\sqrt{1-e^{-2\lambda t}} - 1| = \frac{e^{-2\lambda t}}{\sqrt{1-e^{-2\lambda t}}+1}, \quad \text{morally, one needs to solve}$$

$$\left\| \mathcal{N}\left(\sqrt{\frac{2\lambda}{\epsilon}}e^{-\lambda t}x_0, 1\right) - \mathcal{N}(0,1) \right\|_{\mathbb{TV}} = \frac{2}{\sqrt{2\pi}} \int_0^{\sqrt{\frac{\lambda}{2\epsilon}}e^{-\lambda t}|x_0|} e^{-z^2/2}\mathrm{d}z \leq \eta,$$

which for t large yields

$$\frac{2}{\sqrt{2\pi}}\sqrt{\frac{\lambda}{2\epsilon}}e^{-\lambda t}|x_0| = \sqrt{\frac{\lambda}{\pi\epsilon}}e^{-\lambda t}|x_0| \leq \eta.$$

OUP Wasserstein mixing time

To compute the η-mixing time, one needs to find the smallest $t > 0$ satisfying

$$\mathcal{W}_2(X_t(x_0), \mu) = \mathcal{W}_2\left(\mathcal{N}\left(e^{-\lambda t}x_0, \frac{\epsilon}{2\lambda}(1-e^{-2\lambda t})\right), \mathcal{N}\left(0, \frac{\epsilon}{2\lambda}\right)\right)$$
$$= \sqrt{e^{-2\lambda t}x_0^2 + \frac{\epsilon}{2\lambda}\left(\sqrt{1-e^{-2\lambda t}}-1\right)^2} \leq \eta.$$

Since

$$\left(\sqrt{1-e^{-2\lambda t}}-1\right)^2 = \frac{e^{-4\lambda t}}{(\sqrt{1-e^{-2\lambda t}}+1)^2},$$

morally, we have

$$e^{-\lambda t}|x_0| \leq \eta.$$

Cutoff phenomenon (in total variation distance)

In the eighties, P. Diaconis and M. Shahshahani asked:

- How many shufflings do we need for the deck of cards to be well-mixed?
- When is it random enough?

Behavior

Table 1. Distance to stationarity for repeated shuffles of 52 cards

	k									
	1	2	3	4	5	6	7	8	9	10
$\|P^k - \pi\|$	1.000	1.000	1.000	1.000	0.924	0.624	0.312	0.161	0.083	0.041

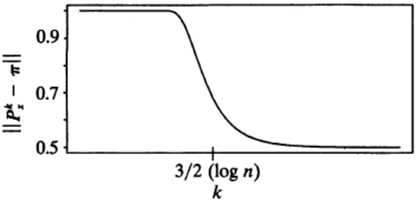


FIG. 1. The cutoff phenomenon for repeated riffle shuffles of n = 52 cards.

Figure: P. Diaconis. *The cut-off phenomenon in finite Markov chains*, PNAS, USA, 93, 1659-1664, 1996.

Random transpositions

- **Space of state**:
  S_n = set of biyections (permutations) on $\{1, 2, \ldots, n\}$. Note that $|S_n| = n!$ is too big.

$$|S_{52}| \gg \text{Avogadro's number} = 6.022 \times 10^{23}.$$

- **Dynamics**: Pairs of cards are exchanged randomly.
- **Equilibrium**: Uniform probability on S_n, that is,

$$\mathbb{P}(\sigma) = \frac{1}{n!} \quad \text{for each permutation} \quad \sigma \in S_n.$$

- **Well-mixed**: $\frac{1}{2} n \log(n)$ steps. The proof can be done by Fourier analysis and representation theory.

The model: unidimensional Langevin dynamics

$$\mathrm{d}X_t^\epsilon = -V'(X_t^\epsilon)\mathrm{d}t + \sqrt{\epsilon}\mathrm{d}W_t, \qquad X_0^\epsilon = x_0,$$

where

i) $V : \mathbb{R} \to [0, \infty)$ is a smooth convex degenerate potential,

ii) $\epsilon \in (0, 1]$ controls the amplitude of the noise,

iii) $W = (W_t)_{t \geq 0}$ is a unidimensional standard Brownian motion,

iv) x_0 is an initial datum.

Limiting distribution

- Under a suitable growth condition on V, the Langevin system possesses a unique invariant probability measure μ^ϵ.
- The density $\mu^\epsilon(\mathrm{d}z)$ is given by

$$\mu^\epsilon(\mathrm{d}z) = \frac{1}{\mathcal{Z}^\epsilon} \exp\left(-\frac{2V(z)}{\epsilon}\right) \mathrm{d}z,$$

  where $\mathcal{Z}^\epsilon$ is the partition function.
- **Weak ergodicity:** For any initial condition x_0, the law of the marginal $X_t^\epsilon(x_0)$ converges in distribution to μ^ϵ as $t \to \infty$, i.e.,

$$\lim_{t\to\infty} \mathbb{E}\left[f(X_t^\epsilon(x_0))\right] = \int_{\mathbb{R}} f(z)\mu^\epsilon(\mathrm{d}z) \quad \text{for all} \quad f \in \mathcal{C}_b(\mathbb{R},\mathbb{R}).$$

Strong ergodicity

- **Strong ergodicity:** For any initial condition x_0, the law of the marginal $X_t^\epsilon(x_0)$ converges in the total variation distance to μ^ϵ as $t \to \infty$, i.e.,

$$\lim_{t\to\infty} \sup_{|f|_\infty \le 1} \left| \mathbb{E}\left[f(X_t^\epsilon(x_0))\right] - \int_{\mathbb{R}} f(z)\mu^\epsilon(\mathrm{d}z) \right| = 0,$$

  where the supremum is taken over all measurable functions $f : \mathbb{R} \to [-1, 1]$.
- For short, we write $\|X_t^\epsilon(x_0) - \mu^\epsilon\|_{\mathbb{TV}}$ instead of

$$\sup_{|f|_\infty \le 1} \left| \mathbb{E}\left[f(X_t^\epsilon(x_0))\right] - \int_{\mathbb{R}} f(z)\mu^\epsilon(\mathrm{d}z) \right|.$$

Rate of convergence

- **Exponential ergodicity:** For any $\epsilon > 0$ there exists a positive constant δ_ϵ such that for any $x_0 \in \mathbb{R}$, there is a positive constant $C_\epsilon(x_0)$ satisfying

$$\|X_t^\epsilon(x_0) - \mu^\epsilon\|_{\mathbb{TV}} \leq C_\epsilon(x_0)e^{-\delta_\epsilon t} \quad \text{for all} \quad t \geq 0.$$

- $C_\epsilon(x_0)$ is hard to compute and/or estimate.
- δ_ϵ is related with the spectral gap of the generator $\mathcal{L}$ of the Markov process $(X_t^\epsilon(x_0))_{t\geq 0}$.
- Lower bounds for $\|X_t^\epsilon(x_0) - \mu^\epsilon\|_{\mathbb{TV}}$ are also very difficult to obtain.

Properties of the variation distance

i) $[0,\infty) \ni t \mapsto \|X_t^\epsilon(x_0) - \mu^\epsilon\|_{\mathbb{TV}} \in [0,1]$ is non-increasing.

ii) Variational representation:

$$\|X_t^\epsilon(x_0) - \mu^\epsilon\|_{\mathbb{TV}} = \inf_{\pi \text{ Coupling}} \pi(\text{ outside of the diagonal }).$$

iii) Dual representation:

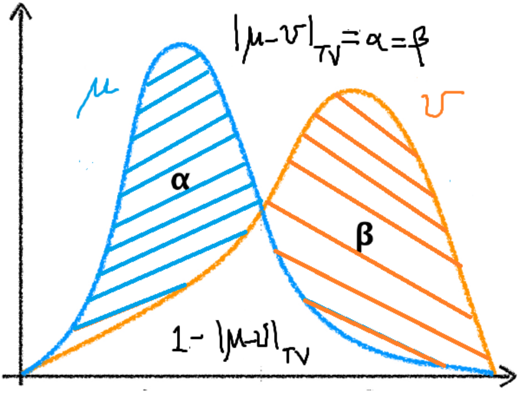
$$\|X_t^\epsilon(x_0) - \mu^\epsilon\|_{\mathbb{TV}} = \sup_{|f|_\infty \leq 1} |\mathbb{E}[f(X_t^\epsilon(x_0))] - \mathbb{E}_{\mu^\epsilon}[f]| \,.$$

iv) Smooth representation:

$$\|X_t^\epsilon(x_0) - \mu^\epsilon\|_{\mathbb{TV}} = \frac{1}{2}\int_{\mathbb{R}} |f^{\epsilon,x_0}(z,t) - f^\epsilon(z)|\mathrm{d}z,$$

where $\mathbb{P}(X_t^\epsilon(x_0) \in \mathrm{d}z) = f^{\epsilon,x_0}(z,t)\mathrm{d}z$ and $\mu^\epsilon(\mathrm{d}z) = f^\epsilon(z)\mathrm{d}z$.

Description of the variation distance



Goal

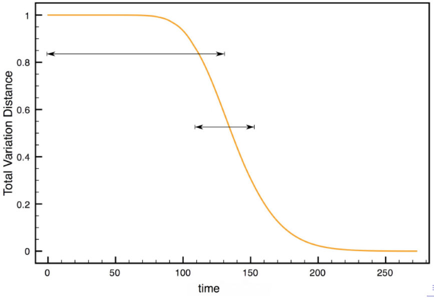
Our aim is to analyze the shape of the convergence to zero of

$$d_t^\epsilon(x_0) = \|X_t^\epsilon(x_0) - \mu^\epsilon\|_{\mathbb{TV}}$$

for each datum $x_0 \in \mathbb{R}$ and degenerate smooth convex potentials V.
In particular, there is cutoff at $(a_\epsilon)_{\epsilon>0}$ if

$$\lim_{\epsilon\to 0^+} d_{a_\epsilon t}^\epsilon(x_0) = \begin{cases} 1 & \text{for} \quad t \in (0,1) \\ 0 & \text{for} \quad t \in (1,+\infty). \end{cases}$$

The cutoff phenomenon is associated to a switching phenomenon.



Stability: degenerate vs hyperbolic fixed points

Let V such that $V'(0) = 0$ and $V''(x) \geq 0$ and consider the IVP

$$\dot{\varphi}_t^{x_0} = -V'(\varphi_t^{x_0}), \qquad \varphi_0^{x_0} = x_0.$$

Degenerate

$$\begin{aligned} V''(0) &= 0, \\ V(x) &= x^4, \\ V(x) &= x^4 + x^6. \end{aligned}$$

Polynomial rate
$|\varphi_t^{x_0}| \sim C_D(x_0) t^{-1/2}$

Hyperbolic

$$\begin{aligned} V''(0) &> 0, \\ V(x) &= x^2, \\ V(x) &= x^2 + x^4. \end{aligned}$$

Exponential rate
$|\varphi_t^{x_0}| \sim C_H(x_0) e^{-2t}$

Theorem (Hyperbolic case, B. & Jara, 2016, JSP)

Let $V:\mathbb{R}\to[0,\infty)$ *be a* $\mathcal{C}^2$*-function with* $V(0)=V'(0)=0$ *satisfying* **the coercivity condition:** *there is a positive* δ *such that* $V''(x)\geq\delta$ *for all* $x\in\mathbb{R}$. *Consider the unique strong solution of the SDE*

$$\begin{cases} \mathrm{d}X_t^\epsilon = -V'(X_t^\epsilon)\mathrm{d}t+\sqrt{\epsilon}\mathrm{d}B_t \quad \text{for} \quad t>0,\\ X_0^\epsilon = x_0, \end{cases}$$

where x_0 *is an initial condition,* $(B_t)_{t\geq 0}$ *is a standard Brownian motion. Then for any* $x_0\neq 0$ *and* $\rho\in\mathbb{R}$ *it follows that*

$$\lim_{\epsilon\to 0^+}\left\|X^\epsilon_{t_\epsilon+\rho\cdot w_\epsilon}(x_0)-\mu^\epsilon\right\|_{\mathbb{TV}}=\frac{2}{\sqrt{2\pi}}\int_0^{\lambda(x_0)e^{-\rho}}\exp\left(-\frac{z^2}{2}\right)\mathrm{d}z,$$

where $t_\epsilon:=\frac{1}{2V''(0)}\ln(1/\epsilon)$, $w_\epsilon:=\frac{1}{V''(0)}+\mathrm{o}_{\epsilon\to 0^+}(1)$,

$$\lambda(x_0):=\frac{\sqrt{2V''(0)}|c(x_0)|}{2}\quad\text{with}\quad c(x_0):=\lim_{t\to\infty}e^{V''(0)t}X_t^0(x_0)>0.$$

Idea for the proof

Linear nonhomogeneous approximation:
Freidlin–Wentzell first order

↓

Cutoff for the nonhomogeneous linear approximation
Hartman–Grobman Theorem

↓

Coupling the nonlinear and linear approximation
Short time coupling

↓

Coupling the limiting measures
Bootstrap argument/Direct computation dimension one

Main result

Theorem (Degenerate case, B., da Costa & Jara, 2022)

Let $V : \mathbb{R} \to [0, \infty)$ *be a* $\mathcal{C}^2$*, convex, even function with* $V(0) = 0$ *that satisfies the following conditions:*

(1) Local behavior at zero: *there exist constants* $C_0 > 0$ *and* $\alpha > 0$ *such that for any* $K > 0$ *we have*

$$\lim_{\lambda \to 0^+} \sup_{|x| \leq K} \left| \frac{V'(\lambda x)}{\lambda^{1+\alpha}} - C_0 |x|^{1+\alpha} \mathrm{sgn}(x) \right| = 0,$$

where $\mathrm{sgn}(x) := {}^{x}/_{|x|}$ *for* $x \neq 0$ *and* $\mathrm{sgn}(0) := 0$.

(2) Growth condition at infinity: *there exist* $c_0, R_0 \in (0, \infty)$*, and* $\beta \in (-1, \infty)$ *such that*

$$V'(x) \geq c_0 x^{1+\beta} \quad \textit{for all} \quad x \geq R_0. \qquad \textbf{(G)}$$

Main result

Theorem (Degenerate case)

Consider the unique strong solution of the SDE

$$\begin{cases} \mathrm{d}X_t^\epsilon = -V'(X_t^\epsilon)\mathrm{d}t + \sqrt{\epsilon}\mathrm{d}B_t \quad \text{for} \quad t > 0, \\ X_0^\epsilon = x_0, \end{cases}$$

where x_0 is an initial condition, $(B_t)_{t\geq 0}$ is a standard Brownian motion. Then for $t > 0$ it follows that

$$\lim_{\epsilon \to 0^+} \left\| X_{a_\epsilon t}^\epsilon(x_0) - \mu^\epsilon \right\|_{\mathbb{TV}} = \| Y_t(x_0) - \nu \|_{\mathbb{TV}} \in (0,1),$$

where $a_\epsilon = \epsilon^{-\frac{\alpha}{2+\alpha}}$, $(Y_t(x_0))_{t\geq 0}$ is the unique solution of the SDE

$$\begin{cases} \mathrm{d}Y_t = -C_0|Y_t|^{1+\alpha}\mathrm{sgn}(Y_t)\mathrm{d}t + \mathrm{d}W_t \quad \text{for} \quad t > 0, \\ Y_0 = \mathrm{sgn}(x_0)\infty, \end{cases} \tag{1}$$

and ν is the unique invariant probability measure for (1).

No cutoff phenomenon

Corollary (No cutoff phenomenon)

With the assumptions and notations of the previous theorem, for any $x_0 \in \mathbb{R}$, the family of processes $(X^\epsilon(x_0))_{\epsilon\in(0,1]}$ does not exhibit cutoff in the total variation distance as $\epsilon \to 0$. In other words, there is no time scale $(t_\epsilon)_{\epsilon\in(0,1]}$ with $t_\epsilon \to \infty$ as $\epsilon \to 0$ and

$$\lim_{\epsilon\to 0^+} \left\| X^\epsilon_{\delta t_\epsilon}(x) - \mu^\epsilon \right\|_{\mathbb{TV}} = \mathbb{1}_{(0,1)}(\delta) \quad \textit{for any} \quad \delta > 0,\ \delta \neq 1.$$

Proof of the previous corollary

Let $A_\epsilon \to \infty$ as $\epsilon \to 0^+$.
i) Assume that $A_\epsilon \le Ca_\epsilon$ for some positive constant C and for all $\epsilon \ll 1$. Then for all $t \ge 0$ we have

$$\left\|X^{\epsilon}_{Ca_\epsilon t}(x_0) - \mu^{\epsilon}\right\|_{\mathbb{TV}} \le \left\|X^{\epsilon}_{A_\epsilon t}(x_0) - \mu^{\epsilon}\right\|_{\mathbb{TV}}$$

and consequently the main theorem for all $t > 0$ yields

$$\begin{aligned} 0 < \|Y_{Ct}(x_0) - \nu\|_{\mathbb{TV}} &= \liminf_{\epsilon \to 0^+} \left\|X^{\epsilon}_{Ca_\epsilon t}(x_0) - \mu^{\epsilon}\right\|_{\mathbb{TV}} \\ &\le \liminf_{\epsilon \to 0^+} \left\|X^{\epsilon}_{A_\epsilon t}(x_0) - \mu^{\epsilon}\right\|_{\mathbb{TV}}. \end{aligned}$$

In particular, for $t > 1$ we have

$$\liminf_{\epsilon \to 0^+} \left\|X^{\epsilon}_{A_\epsilon t}(x_0) - \mu^{\epsilon}\right\|_{\mathbb{TV}} > 0.$$

ii) The case $a_\epsilon \le CA_\epsilon$ for all $\epsilon \ll 1$ is completely analogous using that the limiting profile $\|Y_t(x_0) - \nu\|_{\mathbb{TV}} < 1$ for all $t > 0$.

No cutoff phenomenon via scaling procedure

By the Chapman–Kolmogorov equation, for any $x_0 \in \mathbb{R}$ and $\epsilon \in (0,1]$ it follows that the map

$$t \mapsto d_t^\epsilon(x_0) := \|X_t^\epsilon(x_0) - \mu^\epsilon\|_{\mathbb{TV}} \qquad \text{is non-increasing.}$$

Lemma (Scaling argument)

Let $x_0 \in \mathbb{R}$ and assume that there is a sequence $(a_\epsilon(x_0))_{\epsilon \in (0,1]}$ for which the following conditions hold true:

i) $\lim_{\epsilon \to 0} a_\epsilon(x_0) = \infty$.

ii) *For any $t > 0$*

$$0 < \liminf_{\epsilon \to 0^+} d^\epsilon_{a_\epsilon(x_0)t}(x_0) \leq \limsup_{\epsilon \to 0^+} d^\epsilon_{a_\epsilon(x_0)t}(x_0) < 1.$$

Then there is no cutoff for the family $(X^\epsilon(x_0))_{\epsilon \in (0,1]}$ in total variation distance as ϵ tends to zero.

Mixing times asymptotics

Lemma (Scaling argument)

If in addition, the function

$$t \mapsto d_t^\epsilon(x_0) := \|X_t^\epsilon(x_0) - \mu^\epsilon\|_{\mathbb{TV}} \quad \textit{is continuous and striclty decreasing,}$$

and for all $t > 0$ the limit

$$\lim_{\epsilon \to 0} d^\epsilon_{a_\epsilon(x_0)t}(x_0) = G_{x_0}(t) \in (0,1)$$

then for any $\eta \in (0,1)$ it follows that

$$\lim_{\epsilon \to 0} \frac{\tau_{mix}^{\epsilon, x_0}(\eta)}{a_\epsilon(x_0)} = \inf\{t \geq 0 : G_{x_0}(t) \leq \eta\}.$$

Asymptotics

Corollary (Mixing time asymptotics)

Suppose that all assumptions and notation made in previous theorem hold true. For any $x_0 \in \mathbb{R}$ and $\eta \in (0,1)$, the η-mixing time

$$\tau_{mix}^{\epsilon,x_0}(\eta) := \inf\{t \geq 0 : \|X_t^{\epsilon}(x_0) - \mu^{\epsilon}\|_{\mathbb{TV}} \leq \eta\}$$

satisfies the limiting behavior

$$\lim_{\epsilon \to 0} \frac{\tau_{mix}^{\epsilon,x_0}(\eta)}{a_\epsilon} = \inf\{t \geq 0 : \|Y_t(\mathrm{sgn}(x_0)\infty) - \nu\|_{\mathbb{TV}} \leq \eta\}.$$

That is,

$$\tau_{mix}^{\epsilon,x_0}(\eta) \approx a_\epsilon \cdot \inf\{t \geq 0 : \|Y_t(\mathrm{sgn}(x_0)\infty) - \nu\|_{\mathbb{TV}} \leq \eta\} \quad \textit{for} \quad \epsilon \approx 0,$$

where $a_\epsilon = \epsilon^{-\frac{\alpha}{2+\alpha}}$.

Regularization: exponential convergence

By the growth condition we have that there exist $c > 0$, $R > 1$ and $\beta > -1$ such that

$$xV'(x) \geq c|x|^{2+\beta} \quad \text{for} \quad |x| \geq R.$$

Hence, one can prove that for any $\epsilon > 0$ and $\mathfrak{a} > 0$ there exists positive constants $C_1(\epsilon, \mathfrak{a})$ and $C_2(\epsilon, \mathfrak{a})$ such that for all $x_0 \in \mathbb{R}$ it follows

$$\|X_t^\epsilon(x_0) - \mu^\epsilon\|_{\mathbb{TV}} \leq C_1(\epsilon, \mathfrak{a}) \left(e^{\mathfrak{a}|x_0|} + \int_{\mathbb{R}} e^{\mathfrak{a}|z|} \mu^\epsilon(\mathrm{d}z) \right) e^{-C_2(\epsilon,\mathfrak{a})t}$$

for all $t \geq 0$. Note that $C_1(\epsilon, \mathfrak{a})$ and $C_2(\epsilon, \mathfrak{a})$ also depend on β.

Coming down from infinity: ODE

For simplicity, let us assume that $V(x) = x^4/4$ for all $x \in \mathbb{R}$.
In the sequel, we define the solution of

$$\dot{\varphi}_t = -(\varphi_t)^3 \quad \text{with} \quad \varphi_0 = \infty. \tag{2}$$

Consider the Cauchy problem

$$\dot{\varphi}_t^\ell = -(\varphi_t^\ell)^3 \quad \text{with} \quad \varphi_0^\ell = \ell > 0.$$

- Its explicit solution is given by $\varphi_t^\ell = \sqrt{\frac{1}{2t+\ell^{-2}}}$ for all $t \geq 0$.
- Hence, $\varphi_t^\infty = \sqrt{\frac{1}{2t}}$, $t \geq 0$ solves the differential equation (2).
- $\varphi_t^\infty \in \mathbb{R}$ for any $t > 0$.

Coming down from infinity: SDE

In the sequel, we present the main steps for proving that the following SDE is well-posed:

$$\mathrm{d}Y_t = -(Y_t)^3 \mathrm{d}t + \mathrm{d}W_t \quad \text{for} \quad Y_0 = \infty.$$

Let $(Y_t(\ell), t \geq 0)$ is the unique strong solution of the SDE

$$\mathrm{d}Y_t(\ell) = -(Y_t(\ell))^3 \mathrm{d}t + \mathrm{d}W_t \quad \text{with} \quad Y_0(\ell) = \ell.$$

- **Monotonicity:** Set $Y_t^* = \lim_{\ell \uparrow \infty} Y_t(\ell)$ for $t \geq 0$. Note that $Y_0^* = \infty$.
- **Tightness w.r.t. the i. c.:** $\mathbb{E}[(Y_t(\ell))^2] \leq \psi_t < \infty$ for $\ell, t > 0$ $\underbrace{\Longrightarrow}_{\text{Fatou's Lemma}}$ $Y_t^* \in \mathbb{R}$ for $t > 0 \Longrightarrow$ **Continuous extension**.
- **Markovianity:** $(\mathcal{P}_\ell(\cdot))_{\ell \in [-\infty, \infty]}$.

Scale analysis and limiting shape for a toy model

$V(x) = x^4/4, \quad a_\epsilon = \epsilon^{-1/2}, \quad b_\epsilon = \epsilon^{1/4}, \quad Y_t^\epsilon(x_0) := (1/b_\epsilon) X_{a_\epsilon t}^\epsilon(x_0).$

Goal: analyze

$$d_{a_\epsilon t}^\epsilon(x_0) := \left\| X_{a_\epsilon t}^\epsilon(x_0) - \mu^\epsilon \right\|_{\mathbb{TV}} = \left\| Y_t^\epsilon(x_0) - \mu^1 \right\|_{\mathbb{TV}}.$$

$$\begin{cases} \mathrm{d}Y_t^\epsilon = -(Y_t^\epsilon)^3 \mathrm{d}t + \mathrm{d}W_t, \\ Y_0^\epsilon = x_0/b_\epsilon, \end{cases} \qquad \begin{cases} \mathrm{d}Y_t = -Y_t^3 \mathrm{d}t + \mathrm{d}W_t, \\ Y_0 = \mathrm{sgn}(x_0)\infty. \end{cases}$$

$(Y_t^\epsilon(x_0))_{t\geq 0}$ converges, as $\epsilon \to 0^+$, to $(Y_t(x_0))_{t\geq 0}$.

Hence, for $t > 0$ we deduce

$$\lim_{\epsilon \to 0^+} d_{a_\epsilon t}^\epsilon(x_0) = \left\| Y_t(x_0) - \mu^1 \right\|_{\mathbb{TV}} \in (0, 1).$$

Scheme of the proof

$$\left\| X^{\epsilon}_{a_\epsilon t}(x_0) - \mu^{\epsilon} \right\|_{\mathbb{TV}}$$

$$\Big\downarrow \quad \epsilon \to 0^+$$

$\| Y_t(x_0) - \mu^1 \|_{\mathbb{TV}} \in (0,1) \longrightarrow$ No cutoff at scale $a_\epsilon \to \infty$

$$\Big\downarrow$$

Asymptotics for mixing and no cutoff for any scale A_ϵ

General degenerate potential: main ideas

Let $a_\epsilon > 0$ and $b_\epsilon > 0$ be scaling parameters to be properly fixed. Define

$$Y_t^\epsilon(x_0) := (1/b_\epsilon) X^\epsilon_{a_\epsilon t}(x_0) \quad \text{for} \quad t \geq 0.$$

By the Itô formula we see that

$$\begin{cases} \mathrm{d}Y_t^\epsilon(x_0) = -\frac{a_\epsilon}{b_\epsilon} V'(b_\epsilon Y_t^\epsilon(x_0))\mathrm{d}t + \frac{\sqrt{\epsilon a_\epsilon}}{b_\epsilon}\mathrm{d}W_t \quad \text{for} \quad t \geq 0, \\ Y_0^\epsilon(x_0) = x_0/b_\epsilon, \end{cases}$$

Define a_ϵ and b_ϵ as the unique solution to the system

$$\begin{cases} \frac{\sqrt{\epsilon a_\epsilon}}{b_\epsilon} = 1, \\ a_\epsilon b_\epsilon^\alpha = 1. \end{cases} \quad \text{i.e.} \quad a_\epsilon = \epsilon^{-\frac{\alpha}{2+\alpha}} \quad \text{and} \quad b_\epsilon = \epsilon^{\frac{1}{2+\alpha}}.$$

Hypothesis **(1) Local behavior at zero:** there exist constants $C_0 > 0$ and $\alpha > 0$ such that for any $K > 0$ we have

$$\lim_{\lambda \to 0^+} \sup_{|x| \leq K} \left| \frac{V'(\lambda x)}{\lambda^{1+\alpha}} - C_0 |x|^{1+\alpha} \mathrm{sgn}(x) \right| = 0,$$

where $\mathrm{sgn}(x) := x/|x|$ for $x \neq 0$ and $\mathrm{sgn}(0) := 0$.

General degenerate potential: main ideas

Since the total variation distance is invariant by scaling, we deduce

$$d^{\epsilon}_{a_\epsilon t}(x_0) = \left\|X^{\epsilon}_{a_\epsilon t}(x_0) - \mu^{\epsilon}\right\|_{\mathbb{TV}} = \|Y^{\epsilon}_t(x_0) - b_\epsilon \mu^{\epsilon}(b_\epsilon \mathrm{d}z)\|_{\mathbb{TV}}.$$

By the triangle inequality we have

$$\Big|\underbrace{d^{\epsilon}_{a_\epsilon t}(x_0)}_{\text{Objective}} - \underbrace{\|Y_t(x_0) - \nu\|_{\mathbb{TV}}}_{\text{Limiting profile}}\Big| \leq \underbrace{\|Y^{\epsilon}_t(x_0) - Y_t(x_0)\|_{\mathbb{TV}}}_{\text{Coupling SDEs}} + \underbrace{\|\nu - b_\epsilon \mu^{\epsilon}(b_\epsilon \mathrm{d}z)\|_{\mathbb{TV}}}_{\text{Laplace method}},$$

where $(Y_t(x_0))_{t\geq 0}$ is the solution of the following SDE

$$\begin{cases} \mathrm{d}Y_t(x_0) = -C_0|Y_t(x_0)|^{1+\alpha}\mathrm{sgn}(Y_t)\mathrm{d}t + \mathrm{d}W_t \quad \text{for} \quad t \geq 0, \\ Y_0(x_0) = \mathrm{sgn}(x_0)\infty, \end{cases}$$

where C_0 and α are the positive constants that appears in local assumption, and ν represents the unique invariant probability measure for the preceding random dynamics.

Coupling around the origin

Hypothesis **(2) Growth condition at infinity:** there exist $c_0, R_0 \in (0,\infty)$, and $\beta \in (-1,\infty)$ such that

$$V'(x) \geq c_0 x^{1+\beta} \quad \text{for all} \quad x \geq R_0. \qquad \textbf{(G)}$$

Observe that $\| Y_t^{\epsilon}(x_0) - Y_t(x_0)\|_{\mathbb{TV}}$ is a complicated term. However,

$$\| Y_t^{\epsilon}(x_0) - Y_t(x_0)\|_{\mathbb{TV}} \leq \underbrace{\left\| Y_t^{\epsilon}(x_0) - \widetilde{Y}_t^{\epsilon}(x_0)\right\|_{\mathbb{TV}}}_{\text{Synchronous coupling and maximal inequalities}} + \underbrace{\left\| \widetilde{Y}_t^{\epsilon}(x_0) - Y_t(x_0)\right\|_{\mathbb{TV}}}_{\text{Girsanov coupling or Kabanov's coupling}}.$$

Gaussian-setting: Cameron–Martin–Girsanov's Theorem, Fokker–Planck estimates, Kabanov, Y. et alt. estimates. For instance, it is known that

$$\|X_t(z) - Y_t(z)\|^2_{\mathbb{TV}} \leq 16 \int_0^t \mathbb{E}[|F(X_s(z)) - G(Y_s(z))|^2] \mathrm{d}s,$$

where F and G are the fields for $(X_t(z))_{t\geq 0}$ and $(Y_t(z))_{t\geq 0}$, respectively.

Main references

- G. Barrera, C. da Costa & M. Jara. *Sharp convergence for degenerate Langevin dynamics*. ArXiv:2209.11026
- G. Barrera & M. Jara: **Abrupt convergence of stochastic small perturbations of one dimensional dynamical systems**. Journal of Statistical Physics 2016.
- G. Barrera & M. Jara: **Thermalisation for small random perturbations of dynamical systems**. Annals of Applied Probability 2020.
- G. Barrera & J. C. Pardo. **Cutoff phenomenon for the Ornstein-Uhlenbeck processes driven by Lévy processes**. Electronic Journal of Probability 2020.
- G. Barrera, J. C. Pardo & M. Högele: **The cutoff phenomenon in total variation for nonlinear Langevin systems with small layered stable noise**. Electronic Journal of Probability 2021.
- G. Barrera, J. C. Pardo & M. Högele: **Cutoff thermalization for Ornstein-Uhlenbeck systems with small Lévy noise in the Wasserstein distance**. Journal of Statistical Physics 2021.
- G. Barrera, J. C. Pardo & M. Högele: **The cutoff phenomenon in Wasserstein distance for nonlinear stable Langevin systems with small Lévy noise**. Journal of Dynamics and Differential Equations 2022.