Droplet dynamics in a two-dimensional rarefied gas under Kawasaki dynamics

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Joint work with A. Gaudillière, F. den Hollander, F.R. Nardi, E. Olivieri and E. Scoppola

Analysis and simulations of metastable systems
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- $\mathcal{X}_N = \{\eta \in \mathcal{X}_\beta : \sum_{x\in\Lambda_\beta} \eta(x) = N\}$ set of configurations with N particles

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- Kawasaki dynamics is a continuous time Markov chain $(X(t))_{t \geq 0}$ with state space $\mathcal{X}_N$ and generator

$$(\mathcal{L}f)(\eta) := \sum_{\{x,y\} \in \Lambda_\beta^*} c(x,y,\eta)[f(\eta^{x,y}) - f(\eta)], \qquad \eta \in \mathcal{X}_\beta$$

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  $\Delta < 2U$: arrival of new particles is faster than dissociation of non–protruding particles

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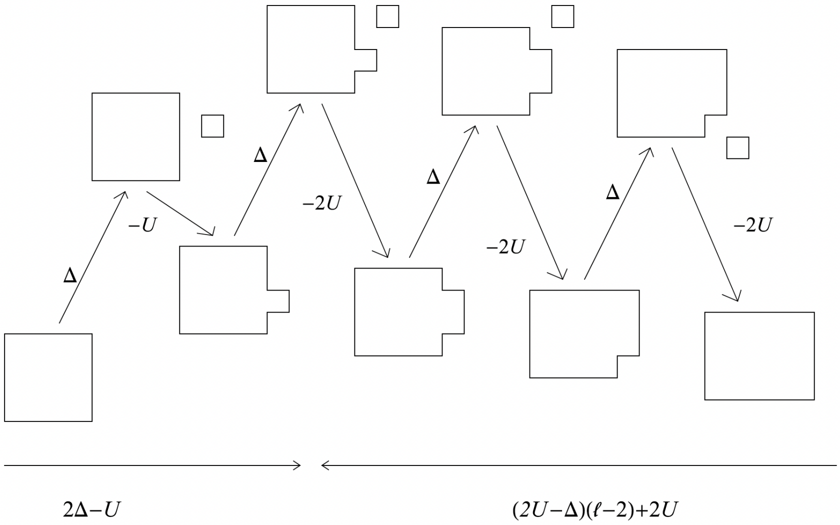

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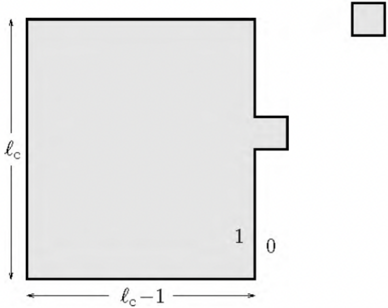

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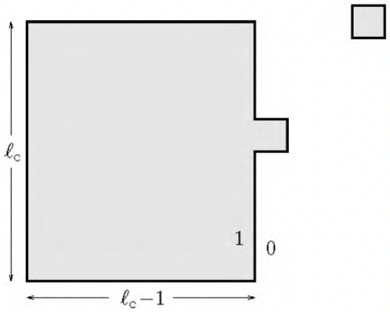

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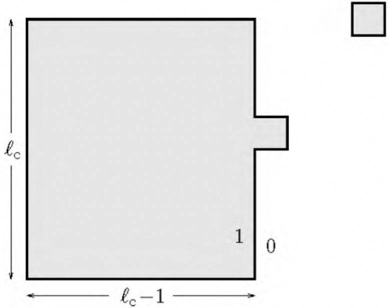

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- for $\ell_1 \geq \ell_c$, the quasi-square $\ell_1 \times \ell_2$ tends to grow

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- a subcritical quasi-square $\ell_1 \times \ell_2$ grows into $\ell_2 \times (\ell_1 + 1)$ with probability larger than

$$e^{-[(2\Delta - U) - r(\ell_1,\ell_2)]\beta - \delta\beta}$$

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- the tube of typical trajectories leading to nucleation is described by a series of events

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- the configurations in moderately large boxes behave as if they are essentially independent and as if the surrounding gas is ideal
- the crossover from the gas to the liquid occurs via homogeneous nucleation: a supercritical quasi-square is created in a moderately large box

Thank you for your attention