Rare transitions in noisy heteroclinic networks

Yuri Bakhtin

Courant Institute, NYU

Analysis and simulations of metastable systems
CIRM, Luminy
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Noisy perturbations of dynamical systems

$$dX = b(X)dt + \varepsilon\sigma(X)dW(t)$$

$$\varepsilon \to 0$$

well-known: Freidlin–Wentzell theory of metastability

transition times between metastable states $\asymp \exp(c_i\varepsilon^{-2})$,

this talk:

a class of models with hierarchy of clusters, transition times $\asymp \varepsilon^{-\theta_i}$

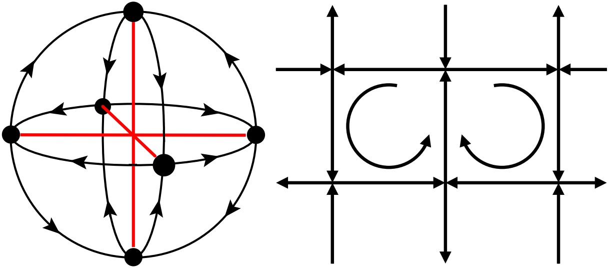
based on preprint:

Yuri Bakhtin, Hong-Bin Chen, Zsolt Pajor-Gyulai, 2022

math arXiv:2205.00326 (141 pp)

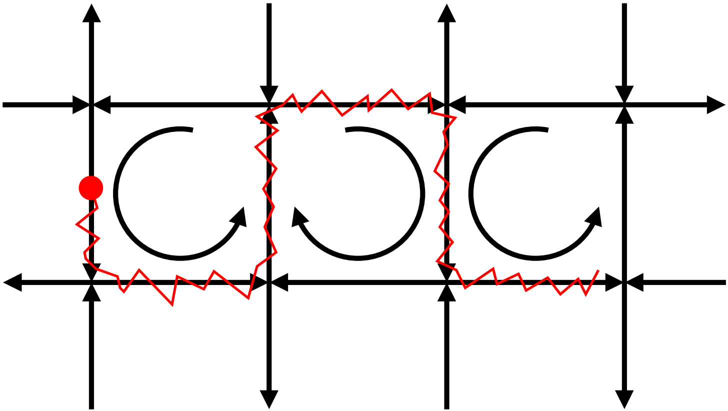
Noisy Heteroclinic Networks

$$dX = b(X)dt + \varepsilon\sigma(X)dW(t)$$



- Neural models, “sequential decision making”, Lotka–Volterra type [Rabinovich, Huerta, Afraimovich 2006]
- Replicator models in biology
- Some gradient flows
- In high dimensions structurally stable under certain conditions [Guckenheimer & Holmes 1988; Krupa 1997]

Small ε

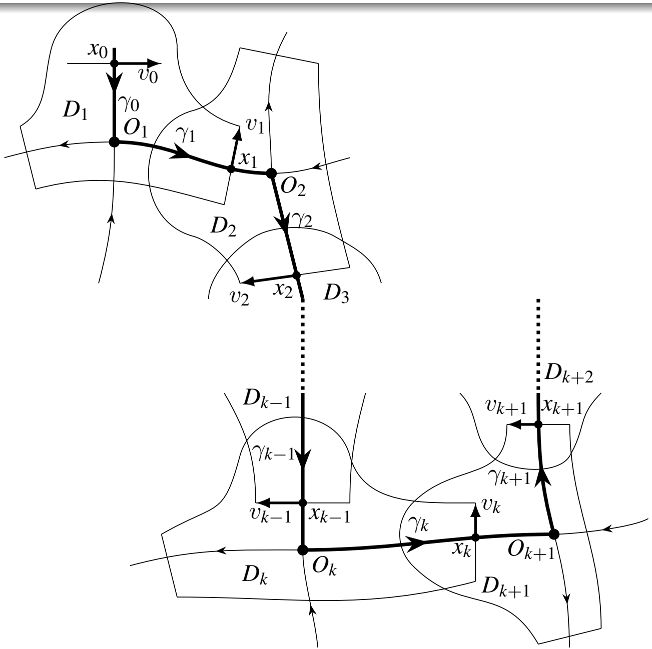


Turns out: some paths are much more frequent than others

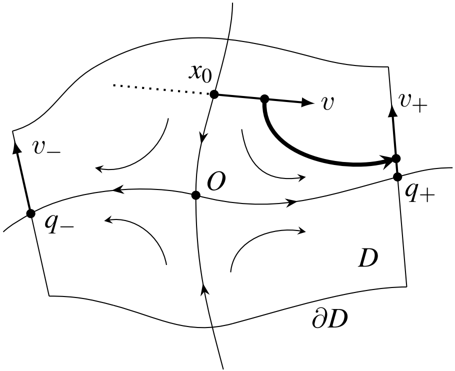
Nonrigorous approach: [Stone,Holmes 1990], [Stone, Armbruster 1999]

Rigorous limit based on asymptotics of random Poincaré maps:
[Bakhtin 2010, 2011]

Sequential exit problems near a heteroclinic chain



Typical exit near one saddle, $\varepsilon \to 0$



Let

$$X(0) = x_0 + \varepsilon^{\alpha} \xi_\varepsilon v$$

$$\alpha \in (0, 1]$$

$$\xi_\varepsilon \xrightarrow{d} \xi, \quad \varepsilon \to 0$$

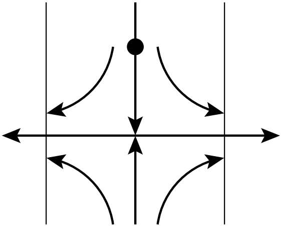
$$\tau = \text{exit time for } D$$

Then

$$X(\tau) = \begin{cases} q_+ + \varepsilon^{\alpha'} \eta_\varepsilon v_+ & \text{with probability converging to } p_+ \\ q_- + \varepsilon^{\alpha'} \eta_\varepsilon v_- & \text{with probability converging to } p_- \end{cases}$$

$$p_+ + p_- = 1, \quad \alpha' \in (0, 1], \quad \eta_\varepsilon \xrightarrow{d} \eta$$

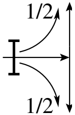
Typical exit near one saddle, $\varepsilon \to 0$



$$dX = \lambda X dt + \varepsilon dW$$
$$dY = -\mu Y dt + \varepsilon dB$$
$$\lambda, \mu > 0 \qquad \rho := \frac{\mu}{\lambda}$$
$$X(0) = \varepsilon^{\alpha}\xi_{\varepsilon} \quad (\xi_{\varepsilon} \xrightarrow{d} \xi), \quad Y(0) = 0$$

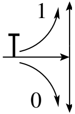
"strong" contraction
$\alpha\rho > 1$

$Y(\tau) = \varepsilon \eta_{\varepsilon}$
$\eta_{\varepsilon} \xrightarrow{d} \eta$ symmetric



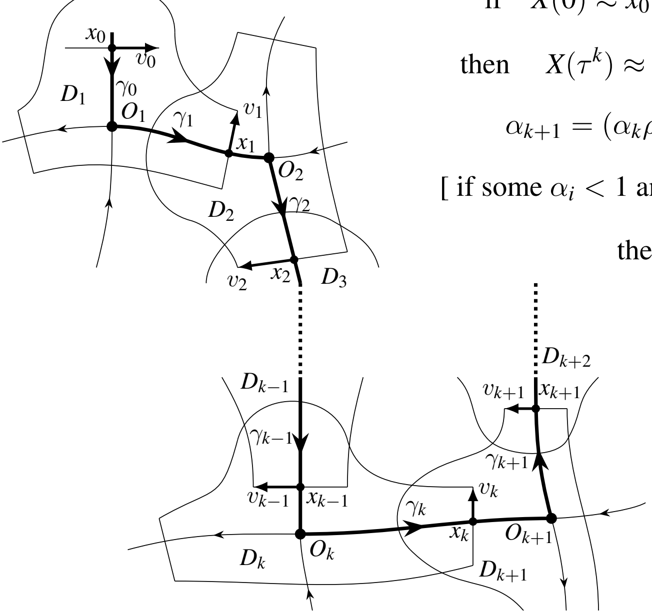
"weak" contraction
$\alpha\rho < 1$

$Y(\tau) = \varepsilon^{\alpha\rho} \eta_{\varepsilon}$
$\eta_{\varepsilon} \xrightarrow{d} \eta > 0$



$$\mathsf{P}_{\varepsilon^{\alpha\rho}\eta}\Big\{\text{exit } \textit{down} \text{ at next saddle}\Big\} \sim \mathsf{P}\Big\{\varepsilon^{\alpha\rho}\eta_{\varepsilon} + \varepsilon \cdot \textit{noise} < 0\Big\} \ll 1$$

Sequential analysis of typical exits, $\varepsilon \to 0$



if $\quad X(0) \approx x_0 + \varepsilon^{\alpha_0}\xi_0 v_0$

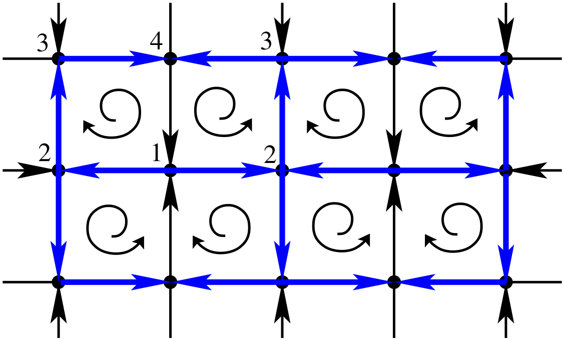
then $\quad X(\tau^k) \approx x_k + \varepsilon^{\alpha_k}\xi_k v_k$

$$\alpha_{k+1} = (\alpha_k \rho_{k+1}) \wedge 1$$

[if some $\alpha_i < 1$ and wrong turn,

then $P \to 0$]

Example: cellular flow



Assumptions

$\rho_1 < 1,\ \rho_2 < 1,\ \rho_3 < 1$

$\rho_1\,\rho_2\,\rho_3\,\rho_4 > 1$

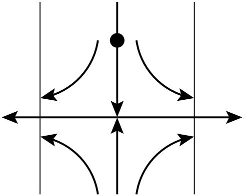
Stability due to strong contraction

at saddle 4;

At times $\sim \log \varepsilon^{-1}$: only circulation within blue cells (pairs of squares).

Longer time scales? Invariant distributions?
Quantify rare transitions?

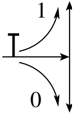
Quantify rare transitions?



$$dX = \lambda X dt + \varepsilon dW$$
$$dY = -\mu Y dt + \varepsilon dB$$
$$\lambda, \mu > 0 \qquad \rho := \frac{\mu}{\lambda}$$
$$X(0) = \varepsilon^{\alpha} \xi_{\varepsilon}$$

“weak” contraction
$\alpha\rho < 1$

$Y(\tau) \approx \varepsilon^{\alpha\rho}\eta_{\varepsilon}$,
$\eta_{\varepsilon} \xrightarrow{d} \eta > 0$

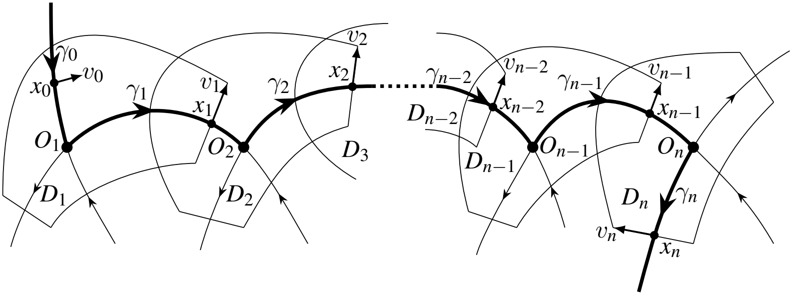


$$\mathsf{P}_{\varepsilon^{\alpha\rho}\eta}\Big\{\text{exit } \textit{down} \text{ at next saddle}\Big\} \sim \mathsf{P}\Big\{\varepsilon^{\alpha\rho}\eta_{\varepsilon} + \varepsilon \cdot \textit{noise} < 0\Big\}$$

Exit *down* at next saddle has a chance to happen only if
$X^2(\tau) = \varepsilon^{\alpha\rho}\eta_{\varepsilon} \asymp \varepsilon$, i.e., $\eta_{\varepsilon} \asymp \varepsilon^{1-\alpha\rho}$.
Need local limit theorems, stronger than weak convergence:

$$\lim_{\varepsilon \to 0} \mathsf{P}\{\eta_{\varepsilon} \in \varepsilon^{1-\alpha\rho}[a, b]\} = ?$$

Cell escape heteroclinic chains of arbitrary length



Theorem (Yuri Bakhtin, Hong-Bin Chen, Zsolt Pajor-Gyulai, 2022)

One of the following holds:

- $P_\varepsilon \to c > 0$
- $P_\varepsilon = h\varepsilon^\theta(1 + o(1)), \quad h, \theta > 0$
- P_ε decays faster than any power of ε

[quantifying transitions between scales, new scalings $\varepsilon^{\bar{\alpha}_i}$, slowdown saddles]

Quantifying transitions between various scales

If the scaling before the saddle is ε^{α},
how likely is the exit at distance $\asymp \varepsilon^{\beta}$?

- If $\alpha < 1$, then exit at scale $\varepsilon^{(\alpha\rho)\wedge 1}$ w.h.p.
- If $\alpha = 1$, $\alpha\rho = \rho \geq 1$, then exit at scale ε^{1} w.h.p.
- If $\alpha = 1$, $\alpha\rho = \rho < 1$, then:
  - exit at ε^{ρ} is typical but
  - for $\beta \in (\rho, 1]$ a local limit theorem holds

$$\mathsf{P}^{\varepsilon x}\{\, X^2(\tau) \in \varepsilon^{\beta}[a,b] \,\} \asymp \varepsilon^{\frac{\beta}{\rho}-1} g(x)\nu([a,b])$$

Proof: Malliavin calculus at growing time scales
Mechanism: longer exposure to contraction

Polynomial time scales

$\mathsf{P}\{$"wrong turn" *aka* "cell escape"$\} \sim \varepsilon^{\theta}$

Associated time scale $\sim \varepsilon^{-\theta}\log\varepsilon^{-1}$

Order all emerging exponents:

$$\theta_1 < \theta_2 < \ldots < \theta_N$$

Define

$$T_{k,\varepsilon} = \varepsilon^{-\theta_k}\log\varepsilon^{-1}$$

For t_ε satisfying

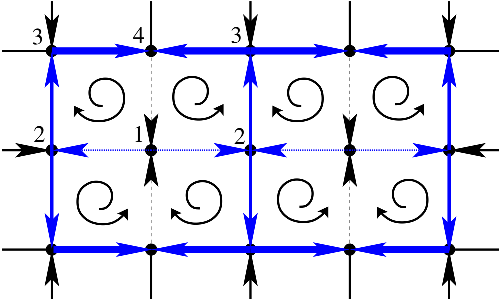
$$T_{k,\varepsilon} \ll t_\varepsilon \ll T_{k+1,\varepsilon},$$

- Many transitions associated with $\theta_1, \theta_2, \ldots \theta_k$
- No transitions associated with $\theta_{k+1}, \theta_{k+2}, \ldots$

Hence

- Hierarchical structure of merging clusters similar to the metastability picture but with polynomial transition times
- In periodic structures of cellular flow type, homogenization results on infinite clusters (CLT)

Back to the cellular flow example



Assumptions

$$\rho_1 < 1,\ \rho_2 < 1,\ \rho_3 < 1$$

$$\rho_1\,\rho_2\,\rho_3\,\rho_4 > 1$$

- Edge 4–1: $\theta = 0$ (clusters: pairs of cells)
- Edge 1–2: $\theta = \dfrac{1}{\rho_1} - 1$ (clusters: squares)
- Edge 2–3: $\theta = \dfrac{1}{\rho_1} - 1 + \dfrac{1}{\rho_2} - 1$ (clusters: strips)
- Edge 3–4: $\theta = \dfrac{1}{\rho_1} - 1 + \dfrac{1}{\rho_2} - 1 + \dfrac{1}{\rho_3} - 1$ (cluster: entire plane)

Actual statements of local theorems in rectified coordinates

10.3. Typical exit locations.

Proposition 10.4. *Let $\alpha \in (0,1]$, $\rho > 0$, $\alpha' = (\alpha\rho) \wedge 1$, $c = R^{-\rho}L$. Let τ be defined by (10.1). Let $\mathcal{U}$ and $\mathcal{N}$ be centered independent Gaussian r.v.'s with variance $\mathbf{c}_1$ and $\mathbf{c}_2$ given in (9.2) and (10.3), respectively. For $a, b, x \in \mathbb{R}$, set*

$$P_\varepsilon^{\alpha,\rho}(x,[a,b]) = \mathsf{P}^{\varepsilon^\alpha x}\left\{Y_\tau \in \{R\} \times \varepsilon^{\alpha'}[a,b]\right\}.$$

For every $\varkappa, \varkappa' > 0$, the following hold for some $\delta > 0$:

(1) If $\rho < 1$, then

$$\sup_{\substack{x \in K_\varkappa(\varepsilon) \\ [a,b] \subset K_{\varkappa'}(\varepsilon)}} \left| P_\varepsilon^{\alpha,\rho}(x,[a,b]) - \mathsf{P}\left\{c\left|x + \varepsilon^{1-\alpha}\mathcal{U}\right|^\rho \in [a,b],\ x + \varepsilon^{1-\alpha}\mathcal{U} \geq 0\right\}\right| = o\left(\varepsilon^\delta\right).$$

(2) If $\rho = 1$, then

$$\sup_{\substack{x \in K_\varkappa(\varepsilon) \\ [a,b] \subset K_{\varkappa'}(\varepsilon)}} \left| P_\varepsilon^{\alpha,\rho}(x,[a,b]) - \mathsf{P}\left\{c\left|x + \varepsilon^{1-\alpha}\mathcal{U}\right| + \varepsilon^{1-\alpha}\mathcal{N} \in [a,b],\ x + \varepsilon^{1-\alpha}\mathcal{U} \geq 0\right\}\right| = o\left(\varepsilon^\delta\right).$$

(3) If $\rho > 1$ and $\alpha\rho \leq 1$, then

$$\sup_{\substack{x \in K_\varkappa(\varepsilon) \\ [a,b] \subset K_{\varkappa'}(\varepsilon)}} \left| P_\varepsilon^{\alpha,\rho}(x,[a,b]) - \mathsf{P}\left\{c|x|^\rho + \varepsilon^{1-\alpha\rho}\mathcal{N} \in [a,b]\right\}\mathsf{P}\left\{x + \varepsilon^{1-\alpha}\mathcal{U} \geq 0\right\}\right| = o\left(\varepsilon^\delta\right).$$

(4) If $\rho > 1$ and $\alpha\rho > 1$, then

$$\sup_{\substack{x \in K_\varkappa(\varepsilon) \\ [a,b] \subset K_{\varkappa'}(\varepsilon)}} \left| P_\varepsilon^{\alpha,\rho}(x,[a,b]) - \mathsf{P}\left\{\mathcal{N} \in [a,b]\right\}\mathsf{P}\left\{x + \varepsilon^{1-\alpha}\mathcal{U} \geq 0\right\}\right| = o\left(\varepsilon^\delta\right).$$

Actual statements of local theorems in rectified coordinates

Proposition 10.2. *Suppose* $\rho < 1$. *Let* $\varkappa, \varkappa' > 0$. *Let* τ *be defined by* (10.1), *and* $\mathbf{c}_1$ *by* (9.2). *Then for each* $\beta \in (\rho, 1)$, *there is* $\delta > 0$ *such that*

$$\sup_{\substack{x \in K_{\varkappa}(\varepsilon) \\ [a,b] \subset K_{\varkappa'}(\varepsilon)}} \left| \varepsilon^{-(\frac{\beta}{\rho}-1)} \mathsf{P}^{\varepsilon x} \left\{ Y_\tau \in \{R\} \times \varepsilon^{\beta}[a,b] \right\} - RL^{-\frac{1}{\rho}} g_{\mathbf{c}_1}(x) \left(|b \vee 0|^{\frac{1}{\rho}} - |a \vee 0|^{\frac{1}{\rho}} \right) \right| = o\left(\varepsilon^{\delta}\right).$$

Proposition 10.3. *Suppose* $\rho < 1$. *Let* $\varkappa, \varkappa' > 0$. *Let* τ *be given in* (10.1), $\mathbf{c}_1$ *in* (9.2), $\mathbf{c}_2$ *in* (10.3). *Then there is* $\delta > 0$ *such that*

$$\sup_{\substack{x \in K_{\varkappa}(\varepsilon) \\ [a,b] \subset K_{\varkappa'}(\varepsilon)}} \left| \varepsilon^{-(\frac{1}{\rho}-1)} \mathsf{P}^{\varepsilon x} \left\{ Y_\tau \in \{R\} \times \varepsilon[a,b] \right\} - RL^{-\frac{1}{\rho}} g_{\mathbf{c}_1}(x)\, \mathsf{E} h(a,b;\mathcal{N}) \right| = o\left(\varepsilon^{\delta}\right)$$

where

$$h(a,b;z) = |(b-z) \vee 0|^{\frac{1}{\rho}} - |(a-z) \vee 0|^{\frac{1}{\rho}} \tag{10.4}$$

and $\mathcal{N}$ *is a centered Gaussian r.v. with variance* $\mathbf{c}_2$.