Subshifts, the emptiness problem and Lovász local lemma

Nathalie Aubrun

(CNRS, Univ. Paris-Saclay, LISN)

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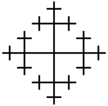
Colorings and subshifts

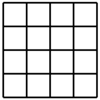
Colorings and subshifts

A finite alphabet, G a group, colorings $x \in A^G$ that avoid some patterns F

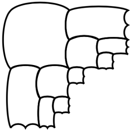
$\langle a, b | \varnothing \rangle$

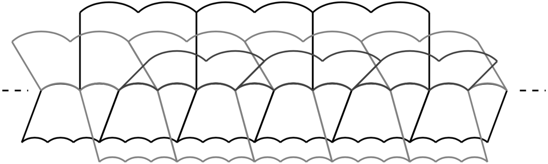


$\langle a, b | ab = ba \rangle$



$\langle a, b | ab^2 = ba^2 \rangle$





$\langle a, b | a^2 b = ba^3 \rangle$

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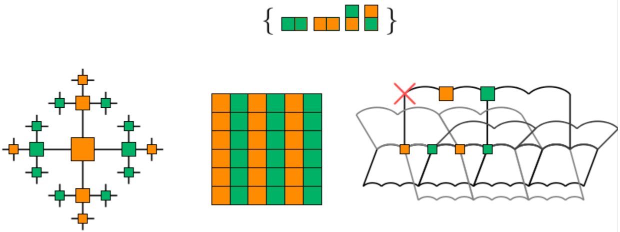

$$X_F := \left\{ x \in A^G \mid \text{ no pattern from } F \text{ appears in } x \right\}$$

Why subshifts?

- Subshifts are **symbolic encodings** of discrete dynamical systems
- SFT (on $\mathbb{Z}^2$): **computation models**
- Space time diagrams of **cellular automata** on $G \approx G \times \mathbb{Z}$ SFTs
- Examples of SFTs: Ice-model, vertex models, ... in **statistical physics**
- ...

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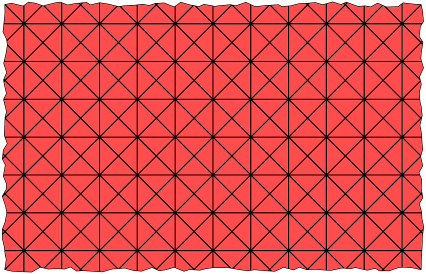

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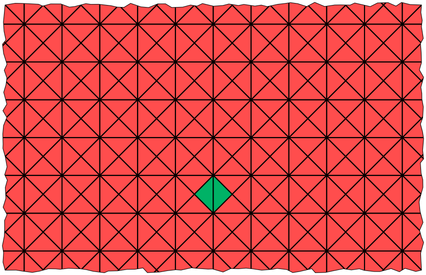

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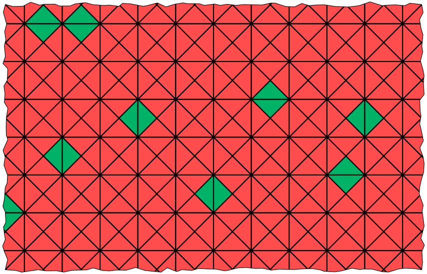

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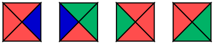

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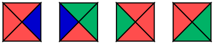

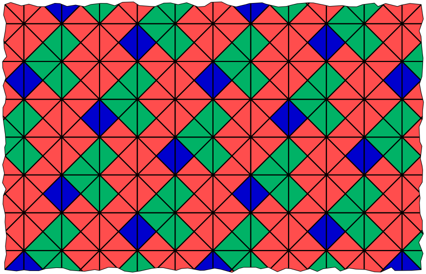

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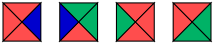

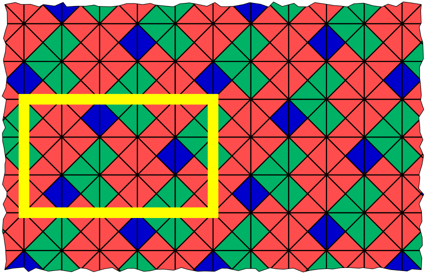

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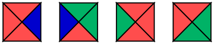

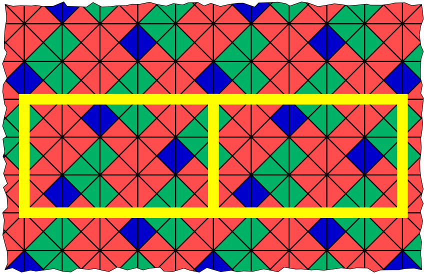

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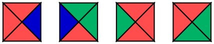

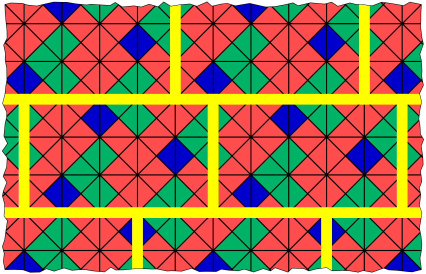

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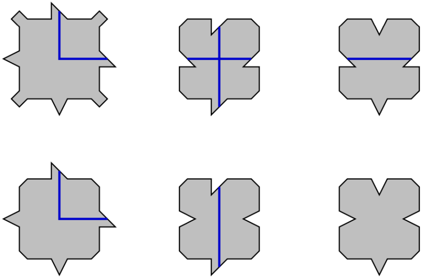

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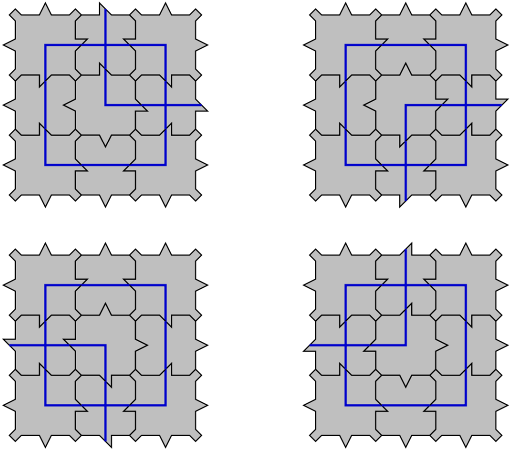

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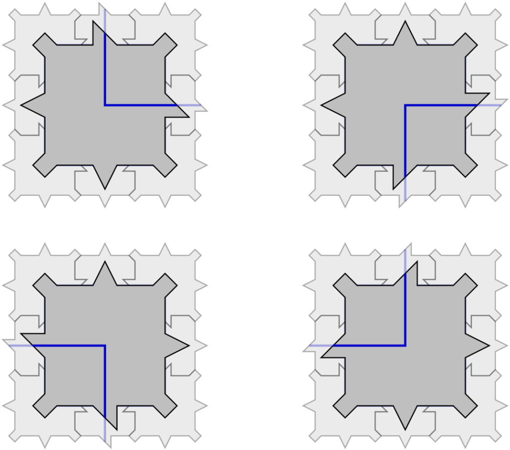

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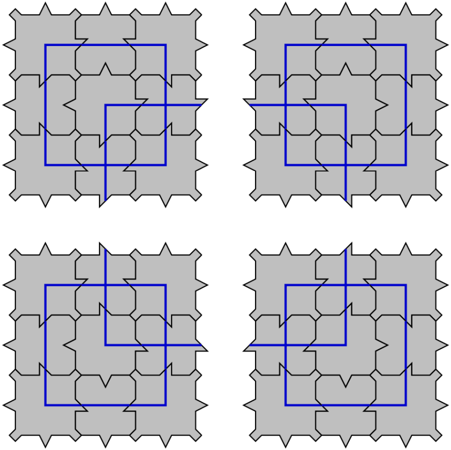

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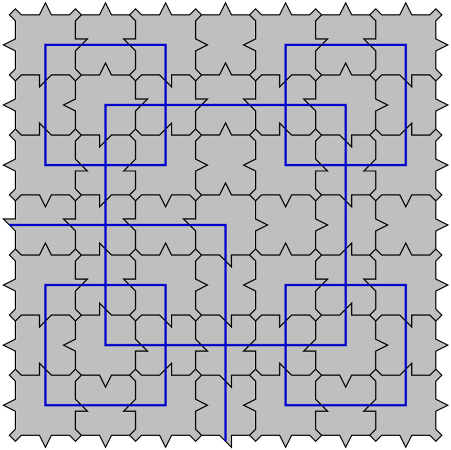

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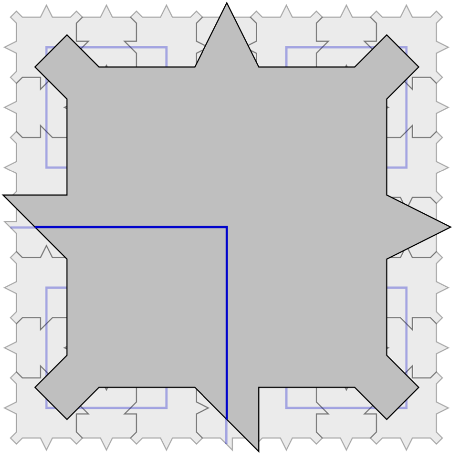

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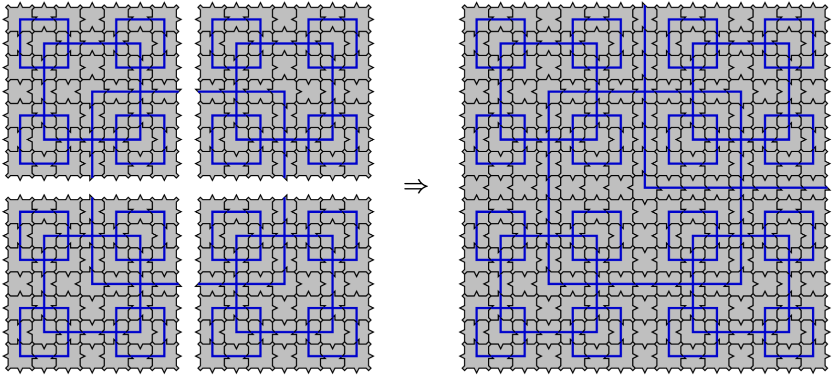

Aperiodic subshifts

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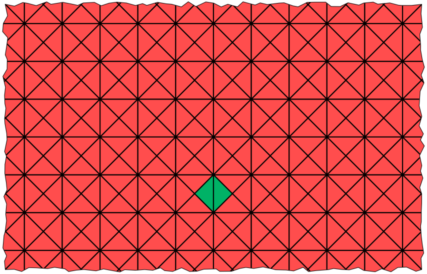

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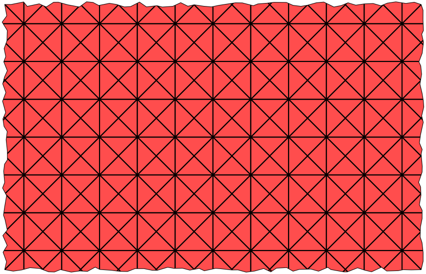

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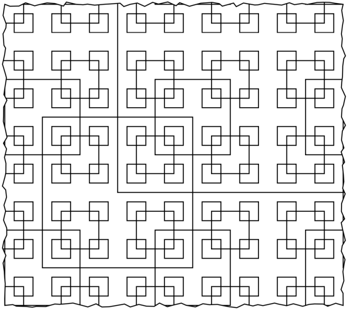

Examples and aperiodicity

Ex.0 Nothing particular. . .

Ex.1 All configurations have finite orbit: **periodic SFT**

Ex.2 All configurations have trivial stabilizer: **strongly aperiodic SFT**

What if we don't know the forbidden patterns?

Prove that a coloring exists from information on the sizes of forbidden patterns

Theorem (Miller, 2009) on $\mathbb{Z}$

Let F be a set of forbidden patterns for $A^{\mathbb{Z}}$. If there exists $c \in (\frac{1}{|A|}, 1)$ s.t.

$$\sum_{f \in F} c^{|f|} \leq c \cdot |A| - 1$$

then X_F is non-empty.

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No hope for a necessary and sufficient condition!

Lovász Local Lemma for subshifts

Lovász Local Lemma

$$\left.\begin{array}{l}(A_i)_{i=1\dots n}\ \textbf{mutually independent}\\ \text{Each } A_i \text{ can be avoided}\end{array}\right\} \Rightarrow A_1, \dots, A_n \text{ can be avoided.}$$

Proposition

If events $A_1, \dots, A_n$ are mutually independent, then

$$Pr\left(\bigcap_{i=1}^{n} \overline{A_i}\right) = \prod_{i=1}^{n} (1 - Pr(A_i)).$$

What about the dependent case ?

Lovász Local Lemma

$$\left.\begin{array}{l}(A_i)_{i=1\dots n} \textbf{\textit{ not very dependent}} \\ \text{Each } A_i \text{ can be avoided}\end{array}\right\} \Rightarrow A_1, \dots, A_n \text{ can be avoided.}$$

Lovász Local Lemma (1975)

Let $E = \{A_1, A_2, \dots, A_n\}$. For $A_i \in E$, let $\Gamma(A_i)$ be the smallest subset of E such that A_i is independent of the collection $E \setminus (\{A_i\} \cup \Gamma(A_i))$. Suppose there are $x_i, \dots, x_n$ such that $0 \le x_i < 1$ and:

$$\forall A_i \in E : Pr(A_i) \le x_i \prod_{A_j \in \Gamma(A)} (1 - x_j)$$

then the probability of avoiding $A_1, A_2, \dots, A_n$ is positive.

$$Pr\left(\bigcap_{i=1}^{n} \overline{A_i}\right) \ge \prod_{A_i \in E} (1 - x_i) > 0.$$

Lovász Local Lemma in Symbolic Dynamics (I)

How to use LLL in Symbolic Dynamics?

Suppose you want to prove that the subshift X is non-empty.

- Uniform Bernoulli measure on configurations space.
- Bad events $\approx$ forbidden patterns.
- Compactness + LLL (if applicable) show the non-emptiness of the subshift.

Lovász Local Lemma in Symbolic Dynamics (II)

Let G be a group, A a finite alphabet and μ the uniform Bernoulli probability measure on A^G.

A sufficient condition for being non-empty

Let $X \subset A^G$ be a subshift defined by $F = \bigcup_{n\geq 1} F_n$, where $F_n \subset A^{S_n}$. Suppose that there exists a function $x : \mathbb{N} \times G \to (0,1)$ such that:

$$\forall n \in \mathbb{N}, g \in G,\ \mu(A_{n,g}) \leq x(n,g) \prod_{\substack{gS_n \cap hS_k \neq \varnothing \\ (k,h) \neq (n,g)}} (1 - x(k,h)),$$

where $A_{n,g} = \left\{ x \in A^G : x|_{gS_n} \in F_n \right\}$. Then the subshift X is non-empty.

Application: strong aperiodicity

Aperiodic SFTs (I)

Examples of strongly aperiodic SFTs on $\mathbb{Z}^2$ (Robinson, Kari-Culik, . . .)

Question

Which groups admit strongly aperiodic SFTs?

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Which groups admit strongly aperiodic SFTs?

- Free groups $\mathbb{F}_n$ do not!
- Generalization of Kari's construction to some $G \times \mathbb{Z}$ (Jeandel, 2015).
- $\mathbb{Z}^n$, **Heisenberg group** (Sahin, Schraudner & Ugarcovici, 2015).
- **Surface groups** (Cohen & Goodman-Strauss, 2015).
- Groups $\mathbb{Z}^2 \rtimes H$ where H has decidable **WP** (Barbieri & Sablik, 2016).
- **Hyperbolic groups** with at most one end (Cohen, Goodman-Strauss & Rieck, 2017).
- **Grigorchuk's group** (Barbieri, 2017).
- **Amenable Baumslag-Solitar groups** (Esnay & Moutot 2022, A. & Kari 2013, A. & Schraudner, 2020).
- **Unimodular Generalized Baumslag-Solitar groups** (A., Bitar & Huriot, 2022+).

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Conjecture (Cohen + Jeandel)

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Difficult problem!!

Two ways to make the question easier:

- characterize groups which admit a ***weakly*** aperiodic SFT?
- characterize groups which admit a strongly aperiodic ***subshift***?

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Idea: express aperiodicity in some relevant way w.r.t. LLL

The distinct neighborhood property

A subshift $X \subset A^G$ is **strongly aperiodic** if all its configurations have trivial stabilizer

$$\forall x \in X, \forall g \in G,\ \sigma^g(x) = x \Rightarrow g = 1_G.$$

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Fix $A = \{0, 1\}$.

A configuration $x \in \{0,1\}^G$ has the **distinct neighborhood property** if for every $h \in G \setminus \{1_G\}$, there exists a finite $T \subset G$ s.t.

$$\forall g \in G,\ x_{|ghT} \neq x_{|gT}.$$

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Proposition (Gao, Jackson & Seward, 2009)

If $x \in \{0,1\}^G$ has the distinct neighborhood property, then the subshift $\overline{Orb_\sigma(x)}$ is strongly aperiodic.

Distinct neighborhood property with LLL

Proposition

Every infinite group G has a configuration $x \in \{0,1\}^G$ with the distinct neighborhood property.

Proof:

- Take $(s_i)_{i\in\mathbb{N}}$ an enumeration of G with $s_0 = 1_G$.
- Choose $(T_i)_{i\in\mathbb{N}}$ a sequence of finite sets of G s.t.

$$T_i \cap s_i T_i = \varnothing \text{ and } |T_i| = Ci \text{ for some constant } C.$$

- $A_{n,g} = \left\{x \in \{0,1\}^G \mid x_{|gT_n} = x_{|gs_nT_n}\right\}$.
- Apply LLL with $x(n,g) = 2^{-\frac{Cn}{2}}$.

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Theorem (A. Barbieri & Thomassé 2019)

Every group G has a strongly aperiodic subshift on alphabet $\{0,1\}$.

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Theorem (A. Barbieri & Thomassé 2019)

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Already proven in (Gao, Jackson & Seward, 2009), but with many pages of descriptive set theory...

A more concrete construction (I)

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Every finite graph with degree $\leq \Delta$ has a square-free coloring with $2e^{16}\Delta^2$ colors.

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Proposition

Let G a f.g. group and S a generating set. Then $\Gamma(G,S)$ has a square-free coloring with $2^{19}|S|^2$ colors.

A more concrete construction (II)

Theorem (A. Barbieri & Thomassé, 2019)

Every group G has an effective strongly aperiodic subshift.

Sketch of the proof:

- Fix S and take $X \subset A^G$ be the subshift such that every square in $\Gamma(G,S)$ is forbidden.
- Let $g \in G$ such that $\sigma^g(x) = x$ for some $x \in X$.
- Factorize g as uwv with $u = v^{-1}$ and $|w|$ minimal (as a word on $(S \cup S^{-1})^*$). If $|w| = 0$, then $g = 1_G$.

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- Factorize g as uwv with $u = v^{-1}$ and $|w|$ minimal (as a word on $(S \cup S^{-1})^*$). If $|w| = 0$, then $g = 1_G$.
- If not, let $w = w_1 \dots w_n$ and consider the odd length walk $\pi = v_0 v_1 \dots v_{2n-1}$ on $\Gamma(G, S)$ defined by:

$$v_i = \begin{cases} 1_G & \text{if } i = 0 \\ w_1 \dots w_i & \text{if } i \in \{1, \dots, n\} \\ w w_1 \dots w_{i-n} & \text{if } i \in \{n+1, \dots, 2n-1\} \end{cases}$$

A more concrete construction (II)

Theorem (A. Barbieri & Thomassé, 2019)

Every group G has an effective strongly aperiodic subshift.

Sketch of the proof:

- Fix S and take $X \subset A^G$ be the subshift such that every square in $\Gamma(G,S)$ is forbidden.
- Let $g \in G$ such that $\sigma^g(x) = x$ for some $x \in X$.
- Factorize g as uwv with $u = v^{-1}$ and $|w|$ minimal (as a word on $(S \cup S^{-1})^*$). If $|w| = 0$, then $g = 1_G$.
- If not, let $w = w_1 \dots w_n$ and consider the odd length walk $\pi = v_0 v_1 \dots v_{2n-1}$ on $\Gamma(G,S)$ defined by:

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- π is a path, and $x_{v_i} = x_{v_{i+n}} \Rightarrow g = 1_G$.

To go further

Other results using LLL in symbolic dynamics to prove the existence of:

- Aperiodic subshift with (any) positive entropy, G amenable/sofic (Bernshteyn, 2019)
- Counting argument $\Rightarrow$ better bounds on the sizes of forbidden patterns (Rosenfeld, 2022+)
- ...

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- Proofs valid for arbitrary groups!
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Thank you for your attention :-)