Synthetic fibered $(\infty, 1)$-category theory

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Joint work with Ulrik Buchholtz: `arXiv:2105.01724`
Thesis at TU Darmstadt supervised by Thomas Streicher

Outline

The concept of $(\infty, 1)$-category

- $(\infty, 1)$**-categories:** “categories weakly enriched in spaces”

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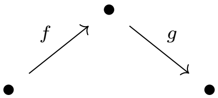

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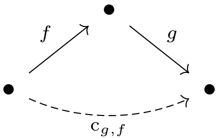

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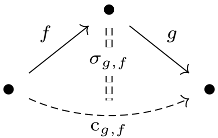

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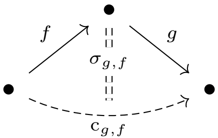

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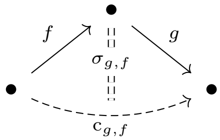

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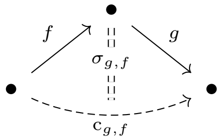

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- Relevant in derived/spectral algebraic geometry, stable homotopy theory, higher algebra, topological field theories, . . .

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- **Take-home slogan:** sHoTT as a convenient, native language (DSL) for Segal objects!

Previous and related work

- **On directed type theory and directed univalence:** Harper–Licata, Warren, Nuyts, Riehl–Shulman, Cavallo–Riehl–Sattler, Weaver–Licata, Bardomiano Martinez. Buchholtz–W, Kudasov, Annenkov–Capriotti–Kraus–Sattler, Finster–Rice–Vicary, Cisinski–Nguyen, North, Altenkirch–Sestini ...
- **On fibrations of $(\infty, 1)$-categories:** Joyal, Lurie, Ayala–Francis, Barwick–Dotto–Glasman–Nardin–Shah, Rasekh, Riehl–Verity ...
- **On Segal spaces and Segal objects/internal $(\infty, 1)$-categories:** Rezk, Joyal–Tierney, Lurie, Kazhdan–Varshavsky, Boavida de Brito, Rasekh, Martini–Wolf ...
- **Proof assistant for sHoTT:** Check out `rzk` developed by Kudasov—prototype interactive proof assistant with online live mode at: `https://github.com/fizruk/rzk`

Outline

sHoTT: Cubes, shapes, and topes

simplicial HoTT [RS17]: Multi-part contexts $\Xi \mid \Phi \mid \Gamma \vdash A$ with pre-type layers[1]

1. **Abstract cubes (*cube layer*):** Lawvere theory generated by directed interval $\mathbb{2}$

$$\frac{}{\mathbf{1}, \mathbb{2}\,\mathrm{cube}} \qquad \frac{}{\Xi \vdash \star : \mathbf{1}} \qquad \frac{}{\Xi \vdash 0, 1 : \mathbb{2}} \qquad \frac{I\,\mathrm{cube} \qquad J\,\mathrm{cube}}{I \times J\,\mathrm{cube}} \qquad \frac{(t : I) \in \Xi}{\Xi \vdash t : I} \qquad [\ldots]$$

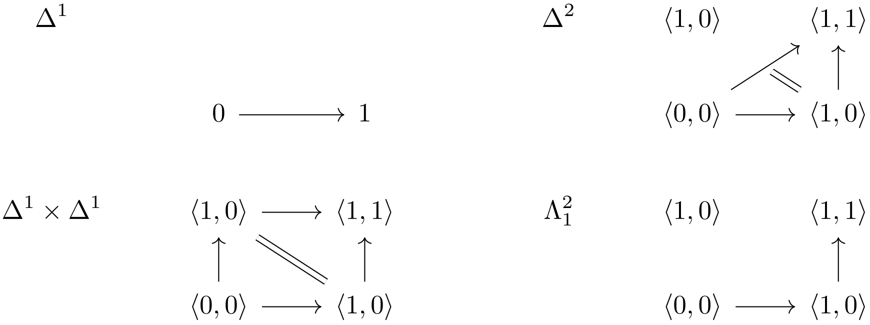
2. **Subpolytopes (*tope layer*):** Intuitionistic theory of formulas φ in cube contexts Ξ

$$\frac{\varphi \in \Phi}{\Xi \mid \Phi \vdash \varphi} \qquad \frac{}{\Xi \vdash \bot, \top\,\mathrm{tope}} \qquad \frac{\Xi \vdash s : I \qquad \Xi \vdash t : I}{\Xi \vdash (s \equiv t)\,\mathrm{tope}} \qquad \frac{\Xi \vdash \varphi\,\mathrm{tope} \qquad \Xi \vdash \psi\,\mathrm{tope}}{\Xi \vdash (\varphi \wedge \psi), (\varphi \vee \psi)\,\mathrm{tope}}$$

$$\frac{}{x, y : \mathbb{2} \vdash (x \leq y)\,\mathrm{tope}} \qquad [\ldots]$$

[1] cf. Cubical Type Theory

sHoTT: Examples of shapes



$$\Delta^1 :\equiv \{t : \mathbb{2} \mid \top\}, \quad \Delta^2 :\equiv \{\langle t, s\rangle : \mathbb{2} \times \mathbb{2} \mid s \leq t\},$$
$$\Delta^1 \times \Delta^1 \equiv \{\langle t, s\rangle : \mathbb{2} \times \mathbb{2} \mid \top\}, \quad \Lambda^2_1 :\equiv \{\langle t, s\rangle : \mathbb{2} \times \mathbb{2} \mid (s \equiv 0) \vee (t \equiv 1)\}$$

sHoTT: Extension types

Idea: “Π-types with strict side conditions”. Originally due to Lumsdaine–Shulman.[2]

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- shape inclusion $\Phi \hookrightarrow \Psi$
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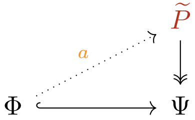
- partial section $a : \Pi_{t:\Phi} P(t)$



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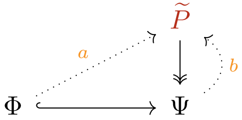

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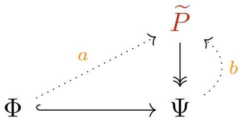
Extension type $\left\langle \prod_\Psi P \big|^{\Phi}_{a} \right\rangle$

with terms $b : \Pi_\Psi P$ such that $b|_\Phi \equiv a$.



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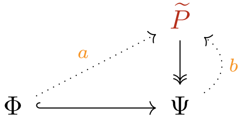

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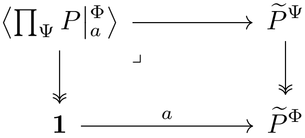
Semantically:



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possesses a diagonal filler uniquely up to contractibility if and only if the following proposition holds:

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$$\mathrm{isContr}\left(\left\langle \Pi_{t:\Psi} P(\sigma(s)) \Big|^{\Phi}_{\kappa} \right\rangle\right)$$

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Hom types I

Definition (Hom types, [RS17])

Let B be a type. Fix terms $a, b : B$. The type of *arrows in* B *from* a *to* b is the extension type

$$\hom_B(a,b) :\equiv (a \to_B b) :\equiv \left\langle \Delta^1 \to B \Big|^{\partial\Delta^1}_{[a,b]} \right\rangle.$$

Definition (Dependent hom types, [RS17])

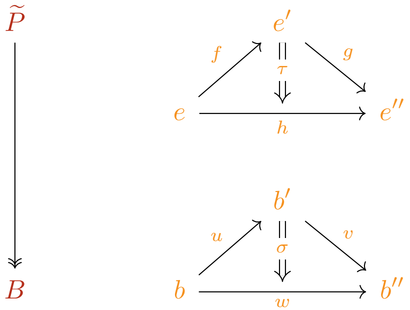
Let $P : B \to \mathcal{U}$ be family. Fix an arrow $u : \hom_B(a,b)$ in B and points $d : P\,a$, $e : P\,b$ in the fibers. The type of *dependent arrows in* P *over* u *from* d *to* e is the extension type

$$\mathrm{dhom}_{P,u}(d,e) :\equiv (d \to^P_u e) :\equiv \left\langle \prod_{t:\Delta^1} P(u(t)) \Big|^{\partial\Delta^1}_{[d,e]} \right\rangle.$$

Hom types II

We will also be considering types of 2-cells: For arrows u, v, w in B with f, g, h in P lying above, with appropriate co-/domains, let

$$\hom^2_B(u,v;w) :\equiv \left\langle \Delta^2 \to B \Big|^{\partial\Delta^2}_{[u,v,w]} \right\rangle, \quad \mathrm{dhom}^{2,P}_\sigma(f,g;h) :\equiv \left\langle \prod_{\langle t,s\rangle:\Delta^2} P(\sigma(t,s)) \Big|^{\partial\Delta^2}_{[f,g,h]} \right\rangle.$$



Segal, Rezk, and discrete(=groupoidal) types

Can now define synthetic ∞-categories[3] using shapes and extension types:

Definition (Synthetic ∞-categories, [RS17])

- **Synthetic pre-∞-category *aka* Segal type:** types A with *weak composition*, *i.e.*:

$$\iota : \Lambda_1^2 \hookrightarrow \Delta^2 \rightsquigarrow A^\iota : A^{\Delta^2} \xrightarrow{\simeq} A^{\Lambda_1^2} \qquad \text{(Joyal)}.$$

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- **Synthetic ∞-groupoid *aka* discrete type:** types A such that *every arrow is invertible*, *i.e.*

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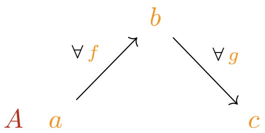
Segal condition

Segal types have *weak composition of morphisms*:

$$\mathrm{isSegal}(A) \simeq \prod_{\kappa:\Lambda_1^2\to A} \mathrm{isContr}\left(\left\langle \Delta^2 \to A\Big|^{\Lambda_1^2}_{\kappa}\right\rangle\right)$$

iff



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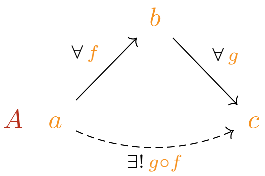

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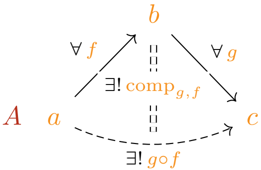

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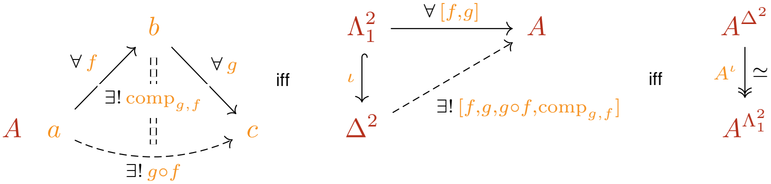

Properties of Segal types

In [RS17] it is shown that:

- Segal types have **categorical structure**: composition $g \circ f$, identities id_x, and homotopies
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- **Closure properties** from orthogonality characterizations, cf. also [BW21]

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Cocartesian families: Motivation

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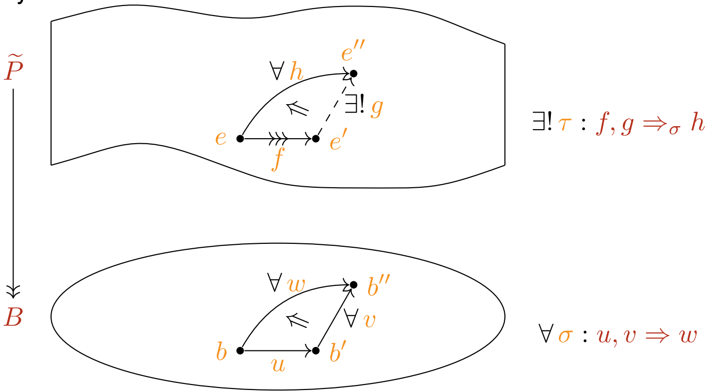
- W [W22]: further extensions to include left exact, bivariant, fibered, and two-sided (*i.e.* mixed-variance) families.
- These are central notions of **fibrations** of synthetic $(\infty, 1)$-categories. They have important applications, and enjoy good properties such as **directed arrow induction** *aka* **type-theoretic Yoneda Lemmas** (originally due to [RS17], also in [RV22]).

Cocartesian arrows: Intuition

Intuitively: An arrow $f : e \to^P_u e'$ over $u : b \to_B b'$ is *cocartesian* if it satisfies the following universal property:



Thanks to Ulrik Buchholtz for the TikZ figures

Cocartesian arrows: Definition

Definition (Cocartesian arrows ([BW21], cf. [RV22]))

Let B be a type and $P : B \to \mathcal{U}$ be an inner family. Let $b, b' : B$, $u : \hom_B(b, b')$, and $e : P\,b$, $e' : P\,b'$. An arrow $f : \hom_{P\,u}(e, e')$ is a *(P-)cocartesian morphism* or *(P-)cocartesian arrow* iff

$$\mathrm{isCocartArr}_P\, f :\equiv \prod_{\sigma : \left\langle \Delta^2 \to B \middle|^{\Delta^1_0}_{u} \right\rangle} \prod_{h : \prod_{t:\Delta^1} P\,\sigma(t,t)} \mathrm{isContr}\left(\left\langle \prod_{\langle t,s\rangle : \Delta^2} P\sigma(t,s) \middle|^{\Lambda^2_0}_{[f,h]} \right\rangle \right).$$

Notice that being a cocartesian arrow is a proposition. Over a Segal base, this amounts to:

$$\mathrm{isCocartArr}_P\, f \simeq \prod_{b'':B} \prod_{v:\hom_B(b',b'')} \prod_{w:\hom_B(b,b'')} \prod_{\sigma:\hom^2_B(u,v;w)} \prod_{e'':P\,b''} \prod_{h:\mathrm{dhom}_{P\,w}(e,e'')} \mathrm{isContr}\left(\sum_{g:\mathrm{dhom}_{P\,v}(e',e'')} \mathrm{dhom}^2_{P\,\sigma}(f,g;h) \right)$$

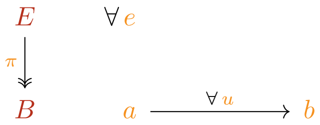
Cocartesian families: Definition

Definition (Cocartesian family ([BW21], cf. [RV22]))

Let B be a Rezk type and $P : B \to \mathcal{U}$ be a family such that $\widetilde{P}$ is a Rezk type. Then P is a *cocartesian family* if:

$$\mathrm{hasCocartLifts}\, P :\equiv \prod_{b,b':B} \prod_{u:b\to b'} \prod_{e:P\,b} \sum_{e':P\,b'} \sum_{f:e\to_u e'} \mathrm{isCocartArr}_P\, f$$

A map $\pi : E \twoheadrightarrow B$ is a *cocartesian fibration* iff $P :\equiv \mathrm{St}_B(\pi)$ is a cocartesian family.



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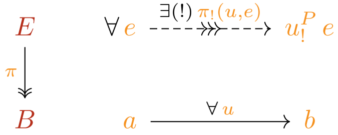

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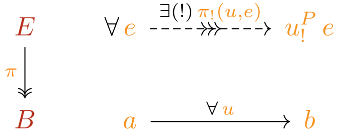

$$\rightsquigarrow (-)_!^P : \prod_{a,b:B} (a \to_B b) \to P(a) \to P(b)$$

Cocartesian families: Functoriality

- Hence, any $u : a \to_B b$ induces a functor $u_! : P\,a \to P\,b$ acting on arrows as follows:

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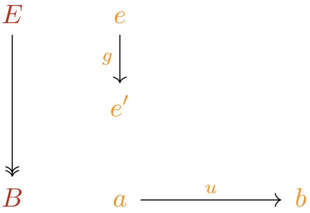

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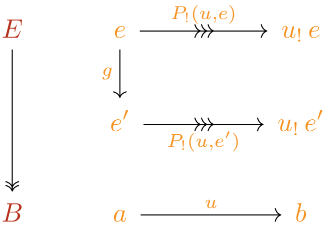

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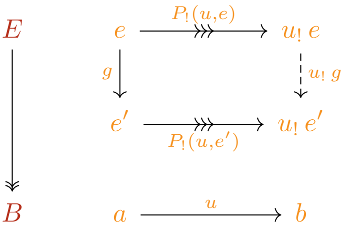

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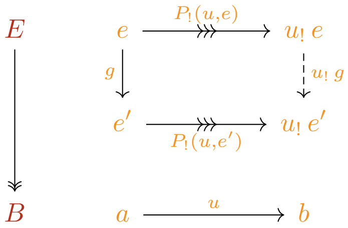

- Externally, this corresponds to a Cat-valued ∞-functor $B \to \mathrm{Cat}$, where Cat is the $(\infty, 1)$-category of small $(\infty, 1)$-categories.

Cocartesian families: Examples

1. For $g : C \to A \leftarrow B : f$, the comma projection $\partial_C : f \downarrow g \twoheadrightarrow C$.[4] (Hence, in particular the codomain projections $\partial_1 : A^{\Delta^1} \twoheadrightarrow A$.)

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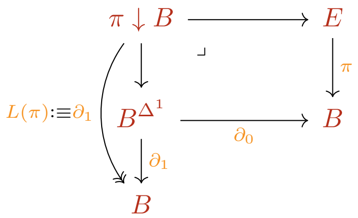
3. For any map $\pi : E \to B$ between Rezk types, the *free cocartesian fibration*:



In particular, the desired UMP holds: $- \circ \iota : \mathrm{CocartFun}_B(L(\pi), \xi) \xrightarrow{\simeq} \mathrm{Fun}_B(\pi, \xi)$ for any cocartesian fibration $\xi : F \to B$.

[4] $f \downarrow g \simeq \Sigma_{b:B, c:C}(f\, b \to_A g\, c)$

Cocartesian families: Characterization

Theorem (Chevalley criterion: Cocartesian families via lifting ([BW21, W22], cf. [RV22]))

Let B be a Rezk type. A given isoinner family $P : B \to \mathcal{U}$ is cocartesian if and only if the Leibniz cotensor map $i_0 \widehat{\pitchfork} \pi : E^{\Delta^1} \to \pi \downarrow B$ has a left adjoint right inverse:

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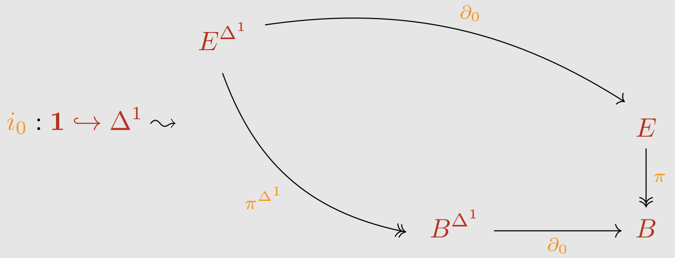

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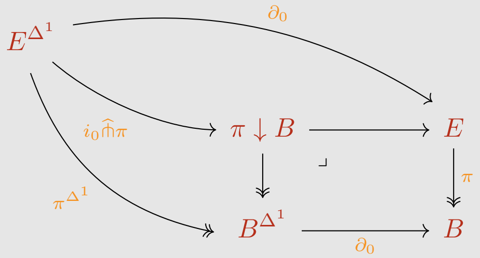

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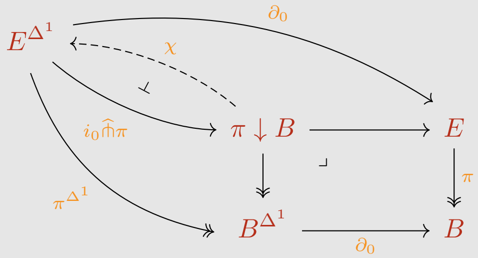

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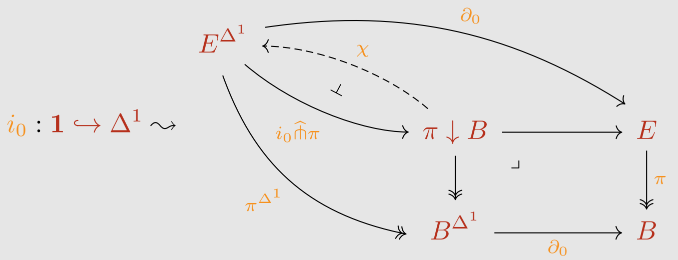

The idea is that $\chi : \pi \downarrow B \to E^{\Delta^1}$ is the **lifting map** $\chi(u, e) = P_!(u, e)$. Chevalley criterion implies a lot of closure properties (cf. ∞-cosmoses)!

Yoneda Lemma for cocartesian families

Theorem (Dependent and absolute Yoneda Lemma ([BW21], cf. [RS17, RV22]))

1. ***Dependent Yoneda Lemma:*** *Let B be a Rezk type, $b : B$ any term, and $Q : b \downarrow B \to \mathcal{U}$ a cocartesian family. Then evaluation at id_b is an equivalence:*

$$\mathrm{ev}_{\mathrm{id}_b} : \prod_{b \downarrow B}^{\mathrm{cocart}} Q \xrightarrow{\simeq} Q(\mathrm{id}_b)$$

2. ***Yoneda Lemma:*** *Let B be a Rezk type, $b : B$ any term, and $P : B \to \mathcal{U}$ a cocartesian family. Then evaluation at id_b as in*

$$\mathrm{ev}_{\mathrm{id}_b} : \prod_{b \downarrow B}^{\mathrm{cocart}} \partial_1^* P \xrightarrow{\simeq} P\, b$$

is an equivalence, where $\partial_1 : b \downarrow B \to B$.

Fibered Yoneda map

Dep. YL is *directed arrow induction*. Its proof uses the following proposition:

Proposition ([BW21], cf. [RS17, RV22])

Let B be a Rezk type, $b : B$ an initial object, and $P : B \to \mathcal{U}$ a cocartesian family. Then evaluation at b given by is an equivalence:

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As a quasi-inverse, we take:

$$\prod_B^{\mathrm{cocart}} P \underset{\mathrm{ev}_b}{\overset{\mathbf{y}}{\rightleftarrows}} P\,b \qquad E \twoheadrightarrow B \qquad d \xrightarrow{P_!(\emptyset_x, d)} \mathbf{y}(d)(x) \text{ over } b \overset{\emptyset_x}{\dashrightarrow} x$$

$$\mathbf{y}(d) :\equiv \lambda x.(\emptyset_x)_!\,d$$

Fibered Yoneda map: valuedness in cocartesian sections

Proposition

The map $\mathbf{y} : P\,b \to \prod_B P$ *is valued in cocartesian sections, i.e. :*

$$\prod_{u:B} \mathrm{isCocartArr}_P((\mathbf{y}d)u)$$

Proof.

See [BW21]. □

Proof of $P\,b \simeq \Pi_B^{\text{cocart}} P$

Proof.

1. 1st roundtrip: $\mathrm{ev}_b(\mathbf{y}d) = \mathbf{y}d(b) = (\mathrm{id}_b)_! d = d$

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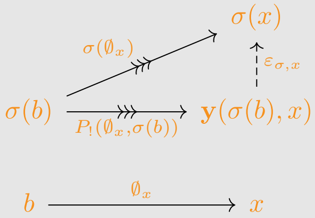

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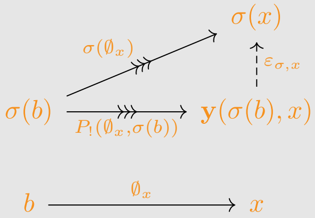


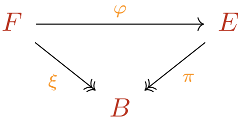
By right cancelation, $\varepsilon_{\sigma,x}$ is cocartesian, too. But then it is an identity.

□

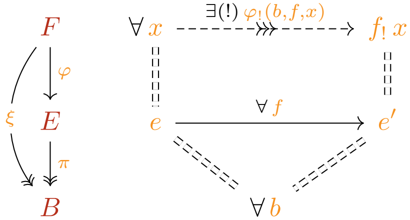
Outline

Sliced cocartesian families

For $\xi : F \twoheadrightarrow B$, $\pi : E \twoheadrightarrow B$, a fibered functor



is a *sliced cocartesian family* ([W22], cf. [RV22]) over B if:



Externally, corresponds to cocartesian fibrations internal to Cat/B (“fibered fibration”).

Two-sided cartesian families

A span

$$A \overset{\xi}{\twoheadleftarrow} E \overset{\pi}{\twoheadrightarrow} B \qquad \leftrightsquigarrow \qquad E \xrightarrow{\langle \xi, \pi \rangle} A \times B$$

is a *two-sided cartesian fibration* ([W22], cf. [RV22]) if

$$\begin{array}{c} E \\ \Big\downarrow {\scriptstyle \langle \xi, \pi \rangle} \\ A \times B \end{array} \qquad \begin{array}{ccccc} v^* \, e & \overset{\exists(!)\ \pi^*(v,e)}{\dashrightarrow} & \forall\, e & \overset{\exists(!)\ \xi_!(u,e)}{\dashrightarrow} & u_! \, e \\ & & & & \\ a & = & a & \xrightarrow{\forall\, u} & a' \\ & & & & \\ b' & \xrightarrow{\forall\, v} & b & = & b \end{array}$$

and the lifts *commute*, *i.e.* canonically

$$u_! \, v^* \, e =_{P(a,b)} v^* \, u_! \, e.$$

Externally, corresponds to ∞-functors $B^{\mathrm{op}} \times A \to \mathrm{Cat}$ ("$(\infty, 1)$-categorical distributors", a kind of higher relation).

Properties of two-sided cartesian families

- **∞-Cosmological closure properties:** By considering two-sided cart. families $P : A \to B \to \mathcal{U}$ as certain "fibered" fibrations, and again using a Chevalley criterion.

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & A \times B \\ & \underset{\pi}{\searrow} \quad \underset{q}{\swarrow} & \\ & B & \end{array}$$

- **(Dependent) Yoneda Lemma for two-sided families:** Let $Q : a \downarrow A \times B \downarrow b \to \mathcal{U}$ be a two-sided family. For $a : A$, $b : B$, evaluation is an equiv.:

$$\mathrm{ev}_{\mathrm{id}_{\langle a,b \rangle}} : \Big(\prod_{a\downarrow A \times B \downarrow b}^{\mathrm{2sCart}} Q \Big) \xrightarrow{\simeq} Q(\mathrm{id}_a, \mathrm{id}_b)$$

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Thank you for your attention!