Groupoidal Realizability

Formalizing the Topological BHK Interpretation

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  - evidence for a conjunction $A \wedge B$ consists of evidence for A together with evidence for B,
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- Exactly what is meant by "evidence" and "process" is unspecified.
- Realizability interpretations are said to formalize the BHK interpretation.
- Example: in Kleene's original *number realizability* evidence comes in the form of natural numbers and processes are partial recursive functions (coded as natural numbers).

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- We are thus led to thinking of types as spaces and higher identities as homotopies.
- Can we develop a realizability interpretation that formalizes the topological BHK interpretation?
- Note: other models could be said to formalize the topological BHK interpretation, eg.:
  - LCCCs that embed **Top** (eg. filter spaces [North '15]),
  - simplicial sets [Voevodsky via Kapulkin-Lumsdaine '12],
  - cubical sets [Cohen-Coquand-Huber-Mörtberg '15, etc.],
  - Quillen model categories (or similar), after [Awodey-Warren '07].

Groupoidal realizability

Idea

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Related work:

- [Hofstra-Warren '12]: combinatorial realizability models.
- Cubical assemblies [Uemura '18, Awodey, Hofstra, Frey, Rosolini].
- [van den Berg '18]: exhibits Hyland's effective topos as the homotopy category of a certain path category.

Outline

1. Groupoidal assemblies
2. A model of type theory
3. Modest groupoids: impredicative and univalent universes

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A morphism $f : (X, A, \Vdash_X^A) \to (Y, B, \Vdash_Y^B)$ is a function $f : X \to Y$ such that $\exists e : A \to B$ satisfying $\forall x \in X$ & $\forall a \in R(A)$:

$$a \Vdash_X^A x \;\Rightarrow\; R(e)(a) \Vdash_Y^B f(x)$$

Realizer categories

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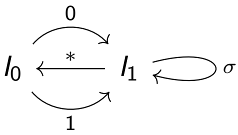
- Q. What sort of thing should realize isomorphisms in a groupoid?
- We ask that $\mathbb{C}$ be weakly cartesian closed and contain an **interval** *qua* co-groupoid $\mathbb{I}$ (with I_0 terminal).



etc.

Example: $I_1 := ([0,1], \{0,1\}) \in \mathbf{Ho}(\mathbf{Top}^2)$.

Realizer categories

- Can elevate $\mathbb{C}$ to a strict ω-category. A **homotopy** $H : f \Rightarrow g : A \to B$ is a map $H : A \times I_1 \to B$ such that:

$$\begin{array}{ccccc} A \times I_0 & \xrightarrow{A\times 0} & A \times I_1 & \xleftarrow{A\times 1} & A \times I_0 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle H} & & \downarrow{\scriptstyle \pi_1} \\ A & \xrightarrow[f]{} & B & \xleftarrow[g]{} & A \end{array}$$

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\end{array}$$

- We also have a **fundamental groupoid functor**:

$$\Pi := (-)^{\mathbb{I}} : \mathbb{C} \to \mathbf{Gpd}$$

The 2-category $\mathbf{GAsm}(\mathbb{C})$ of groupoidal assemblies

- A **groupoidal assembly** is of the form:

$$\left(X \in \mathbf{Gpd}, A \in \mathbb{C}, \Vdash_X^A \nrightarrow \Pi(A) \times X\right)$$

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$$\alpha \Vdash_X^A p \;\Rightarrow\; \Pi(f)(\alpha) := f \circ a \Vdash_Y^B F(p)$$

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- A **2-cell** $\phi : F \Rightarrow G$ is a natural transformation $\phi : X \times \mathbf{2} \to Y$ such that $\exists H : A \times I_1 \to B$ satisfying:

$$\alpha \Vdash^A_X p \;\Rightarrow\; \Pi(H)(\alpha) := H \circ \langle \alpha, I_1 \rangle \Vdash^B_Y \phi(p, \to)$$

Cartesian closure

- Terminal object:

$$\left(\mathbf{1}, I_0, \Vdash_{\mathbf{1}}^{I_0}\right)$$

where $* \Vdash_{\mathbf{1}}^{I_0} *$.

- Binary product of $(X, A, \Vdash_X^A)$ and $(Y, B, \Vdash_X^B)$:

$$\left(X \times Y, A \times B, \Vdash_{X \times Y}^{A \times B}\right)$$

where $\alpha : I_1 \to A \times B \Vdash_{X \times Y}^{A \times B} (p_1, p_2) \iff \pi_i \circ \alpha \Vdash_X^A p_i$.

- Exponential of $(X, A, \Vdash_X^A)$ by $(X, A, \Vdash_X^A)$:

$$\left(\mathrm{Real}(Y^X), B^A, \Vdash^{B^A}\right)$$

where $H : I_1 \to B^A \Vdash^{B^A} \phi \iff \mu(H) \circ \mathrm{swap} : A \times I_1 \to B \Vdash \phi$.

1 Groupoidal assemblies

2 A model of type theory

3 Modest groupoids: impredicative and univalent universes

Uniform families

- A **uniform family of groupoidal assemblies** over $X = (X, A, \Vdash_X^A)$ is a functor

$$\mathbb{X} : X \to \mathbf{GAsm}(\mathbb{C})$$

such that the $\mathbb{X}(x)$ have a common realizing space.

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  such that the $\mathbb{X}(x)$ have a common realizing space.
- An **object** M **of the family** $\mathbb{X}$ consists of:
  - an object $M(x) \in \mathbb{X}(x)$ for every object $x \in X$,
  - a morphism $M(p) : \mathbb{X}(p)(M(x)) \to M(y) \in \mathbb{X}(y)$ for every morphism $p \in X(x, y)$,

  satisfying "dependent functoriality" laws and such that there exists a uniform realizer $f : A \to B$ satisfying $\forall p \in X(x, y).\ \forall \alpha \in \Pi(A)$:

$$\alpha \Vdash_{A,X} p \ \Rightarrow \ \Pi(f)(\alpha) \Vdash_{B,\mathbb{X}(y)} M(p)$$

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- The *uniformity* of realizing spaces and realizers of objects of families means we get away with a simply-typed (CC) realizer category.

Identity types

$$
\begin{aligned}
\mathsf{Id}_X : X \times X &\to \mathbf{GAsm}(\mathbb{C}) \\
(x_1, x_2) &\mapsto \left(X(x_1, x_2), A^{I_1}, \Vdash^{A^{I_1}}_{X(x_1,x_2)} \right)
\end{aligned}
$$

where

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\alpha : I_0 \to A^{I_1} \Vdash^{A^{I_1}}_{X(x_1,x_2)} p \;\Leftrightarrow\; \mu(\alpha) : I_1 \to A \Vdash^{A}_{X} p
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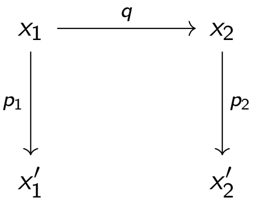

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$$\begin{array}{ccc} x_1 & \xrightarrow{\quad q \quad} & x_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ x_1' & \xrightarrow[\mathrm{Id}_X(p_1,p_2)(q)]{} & x_2' \end{array}$$

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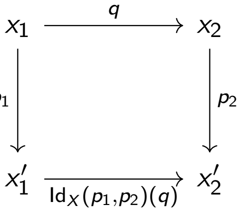
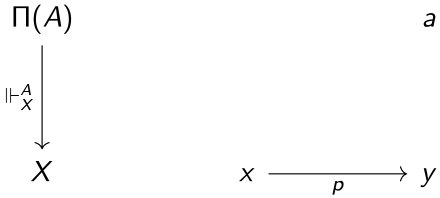


Need a map $A^{I_1} \to A^{I_1}$ that realizes this functor.

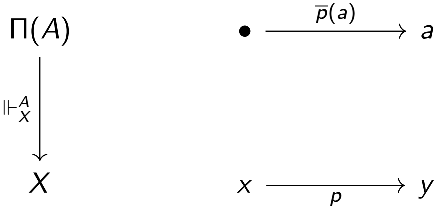
Identity types: fibred realizability (i)

For all $a \in \Pi(A)$ and all $p : x \to y$ such that $a \Vdash_X^A y$



Identity types: fibred realizability (i)

For all $a \in \Pi(A)$ and all $p : x \to y$ such that $a \Vdash^A_X y$ there exists a lift $\overline{p}(a) \in \Pi(A)$ such that $\overline{p}(a) \Vdash^A_X p$.



Identity types: fibred realizability (ii)

- For any $p : x \to y$ there is a map $\mathrm{pre}(p) : A^{I_1} \to A^{I_1}$ that behaves like precomposition with chosen lifts.

$$\begin{array}{c} \Pi(A) \\ \Big\downarrow{\scriptstyle \Vdash_{A,X}} \\ X \end{array} \qquad\qquad \begin{array}{c} a \xrightarrow{\alpha} b \\ \\ \\ x \xrightarrow[p]{} y \xrightarrow[q]{} z \end{array}$$

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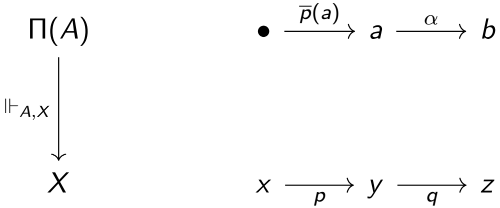

- It is possible to define from this an analogous postcomposition map. Thus the functor $\mathrm{Id}_X(p_1, p_2)$ is realized by $\mathrm{post}(p_2) \circ \mathrm{pre}(p_1^{-1})$.

Example assembly in $\mathbf{GAsm}\left(\mathbf{Ho}(\mathbf{Top}^2)\right)$

- The “circle assembly” whose underlying groupoid is $\mathbb{Z}$.
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- Realizability clauses:

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a &\Vdash_{\mathbb{Z},\mathbb{S}^1} * \\
[\omega_a^z] &\Vdash_{\mathbb{Z},\mathbb{S}^1} z \quad (\omega_a^z = \text{“z-loop” based at } a)
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- Chosen lift at a of z is $[\omega_a^z]$.
- Precomposition operation:

$$\mathrm{pre}(z) : (\mathbb{S}^1)^{I_1} \to (\mathbb{S}^1)^{I_1}$$
$$\mathrm{pre}(z)[\omega_a^{z'}] := [\omega_a^{z+z'}]$$

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- $\Vdash_X^A$ is such a groupoid! But we want a groupoid in the image of Π.

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- So change realizing space from B^A to $B^{\mathcal{S}(\Vdash_X^A)}$.

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Universal objects

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- A realizer category provides a *typed* notion of realizability. To make it untyped (like a PCA) we postulate a "universal object" $U \in \mathbb{C}$ such that every object of $\mathbb{C}$ is a retract of U.

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- The inclusion of the subcategory of **GAsm**($\mathbb{C}$) spanned by the objects with U as the realizing space is an equivalence. Henceforth we work with this subcategory.

Modest groupoids and generalized congruences

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- A groupoidal assembly is **modest** if the objects of its underlying groupoid are uniquely determined by their realizers, and the same for morphisms.
- A family is modest if it factors through modest groupoids. $\mathsf{Id}_{\mathbb{X}}$ is modest when $\mathbb{X}$ is; $\Pi(\mathbb{X}, \mathbb{Y})$ is modest when $\mathbb{Y}$ is.

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- A family is modest if it factors through modest groupoids. $\mathsf{Id}_{\mathbb{X}}$ is modest when $\mathbb{X}$ is; $\Pi(\mathbb{X}, \mathbb{Y})$ is modest when $\mathbb{Y}$ is.
- A **generalized congruence** on a category $\mathbb{D}$ consists of a PERs on $\mathrm{Ob}(\mathbb{D})$ and $\mathrm{Mor}(\mathbb{D})$ that behave nicely together. A **morphism between GCs** on $\Pi(U)$ is a functor between their quotient categories that is "tracked" (realized) by a morphism $U \to U$.

Impredicative universes

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$$\mathcal{U} := \mathrm{Core}\left(\mathbf{GC}\left(\Pi(U)\right)\right)$$

- We make this groupoid into an assembly by saying that all points in $\Pi(U)$ realize every GC and that a morphism $F : \sim \to \sim'$ of GCs is relized by $\alpha : I_1 \to U$ iff $\forall \beta \in \Vdash_{\mathcal{M}(\sim)}$:

$$\Pi\left(\mu(\hat{\alpha})\right)\left(\sigma_{\Vdash_{\mathcal{M}(\sim)}}(\beta)\right) \Vdash_{\mathcal{M}(\sim')} F(\pi_2\beta)$$

where

$$\hat{\alpha} \quad = \quad I_0 \xrightarrow{\lambda(\alpha)} U^{I_1} \xrightarrow{s_{U^{I_1}}} U \xrightarrow{r} \left[\mathcal{S}\left(\Vdash_{\mathcal{M}(\sim)}\right), U\right]$$

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Theorem

Suppose that $\mathbb{C}$ is (strictly) cartesian closed and that the retraction

$$U^{I_1} \lhd U$$

is an isomorphism

$$U^{I_1} \cong U$$

Then $\mathcal{U}_0$ is univalent.

Comparing forms of extensionality

Modelling	Condition	Slogan
untyped λ-calculus with η (equational extensionality of functions)	$U \cong U^U$ in CCC	"Everything is a function."
univalence (universe extensionality) of $\mathcal{U}_0 \in \mathbf{GAsm}(\mathbb{C})$	$U \cong U^{I_1}$ in realizer category $\mathbb{C}$	"Everything is a path."

Back to your roots: combinatory algebras

- Let $\mathcal{A}$ be a combinatory algebra (model of combinatory logic). We can construct its Karoubi envelope (idempotent completion) $\mathbb{K}(\mathcal{A})$: $\mathrm{Ob}(\mathbb{K}(\mathcal{A})) := \{a \in \mathcal{A} \mid a \circ a = a\}$ (where $b \circ a := \lambda^* x.b \cdot (a \cdot x)$) and $\mathbb{K}(\mathcal{A})(a, b) := \{e \in \mathcal{A} \mid e = b \cdot e \cdot a\}$.

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- [Scott '80]: $\mathbb{K}(\mathcal{A})$ is a CCC with universal object $U := \lambda^* x.x$.
- Thus, taking $\mathbb{I}$ to be the discrete interval on U, $\mathcal{U} \in \mathbf{GAsm}(\mathbb{K}(\mathcal{A}))$ is an impredicative universe of modest groupoids.

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- Thus $\mathcal{U}_0 \in \mathbf{GAsm}(\mathbb{K}(\mathcal{A}))$ is an impredicative and univalent universe of modest sets.
- Note that realizers "collapse" for a discrete interval.
- U satisfies propositional resizing when the interval is discrete and lifting operations are identities.

Conclusion

- Groupoidal assemblies generalize the groupoid model (take the trivial realizer category) and also traditional (set-based) realizability (discrete groupoidal assemblies using a discrete interval).
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- Devise an untyped λ-calculus for realizer categories with a universal object (satisfying univalence condition).
- Applications (consistency, independence, conservativity results).
- Can we exact-complete the category of groupoidal assemblies to obtain a realizability elementary (2,1)-topos?
- Is there a tripos-theoretic approach to groupoidal realizability?
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