Fat delta, pseudofunctors and weak units

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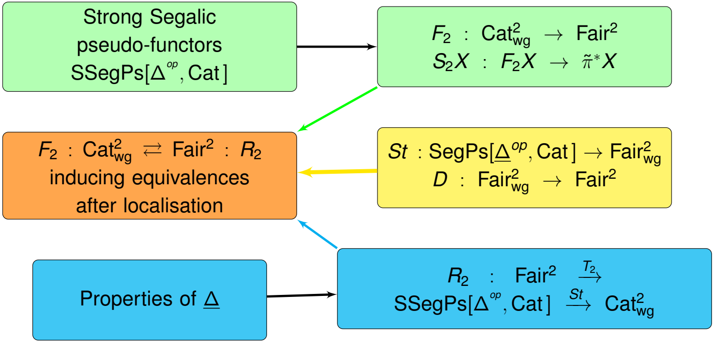
Motivating question

- Fair^2 and $\mathrm{Cat}^2_{\mathrm{wg}}$ are models of weak 2-categories, and are suitable equivalent to bicategories.

- We aim to directly compare Fair^2 and $\mathrm{Cat}^2_{\mathrm{wg}}$, without using the equivalences of Fair^2 and $\mathrm{Cat}^2_{\mathrm{wg}}$ with bicategories.

- This will highlight interesting features of weakly globular double categories and pave the way to higher dimensional generalizations (weak units conjecture).

Overview

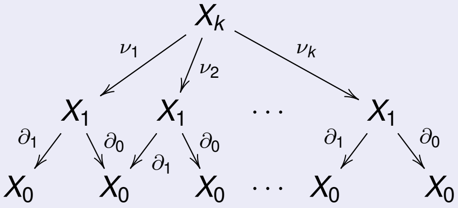


Plan: - Background: the key players, $\mathrm{Cat}^2_{\mathrm{wg}}$, Fair^2, $\mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}]$
- From $\mathrm{Cat}^2_{\mathrm{wg}}$ to Fair^2
- From Fair^2 to $\mathrm{Cat}^2_{\mathrm{wg}}$
- Sketch of proof of main result

Segal maps

Let $X \in [\Delta^{op}, \mathcal{C}]$ be a simplicial object in a category $\mathcal{C}$ with pullbacks. Denote $X[k] = X_k$.

For each $k \geq 2$, let $\nu_i : X_k \to X_1$, $\nu_j = X(r_j)$, $r_j(0) = j - 1$, $r_j(1) = j$



There is a unique map, called Segal map

$$\eta_k : X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \ .$$

- There is a nerve functor

$$N : \mathrm{Cat}\,\mathcal{C} \to [\Delta^{op}, \mathcal{C}]$$

$X \in \mathrm{Cat}\,\mathcal{C}$

$$NX \qquad \cdots X_1 \times_{X_0} X_1 \times_{X_0} X_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; X_1 \times_{X_0} X_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow}\; X_1 \;\substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow}\; X_0$$

Fact: $X \in [\Delta^{op}, \mathcal{C}]$ is the nerve of an internal category in $\mathcal{C}$ if and only if all the Segal maps $\eta_k : X_k \to X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1$ are isomorphisms.

Weakly globular double categories

$X \in [\Delta^{op}, \mathsf{Cat}]$ is in $\mathsf{Cat}^2_{\mathsf{wg}}$ if

i) The Segal maps are isomorphisms:

$$X_k \cong X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \qquad k \geq 2$$

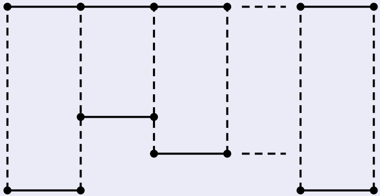
ii) Weak globularity condition: X_0 is an equivalence relation; thus $\gamma : X_0 \to X_0^d$ is an equivalence of categories, where X_0^d is the discrete category on the set of connected components of X_0.

iii) The induced Segal maps are equivalences of categories:

$$X_k \cong X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \xrightarrow{\simeq} X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 \qquad k \geq 2$$

Weak globularity condition

- The set underlying X_0^d plays the role of set of objects.
- The induced Segal map condition is equivalent to



Truncation functor and hom category

- Let $p : \mathrm{Cat} \to \mathrm{Set}$ be the isomorphism classes of objects functor.
- There is a truncation functor

$$p^{(1)} : \mathrm{Cat}^2_{\mathrm{wg}} \to \mathrm{Cat}\,,$$

$(p^{(1)}X)_k = pX_k$ for all $k \geq 0$.

- Given $X \in \mathrm{Cat}^2_{\mathrm{wg}}$, $a, b \in X_0^d$ let $X(a,b)$ be the fibre at (a,b) of

$$X_1 \xrightarrow{(\partial_0,\partial_1)} X_0 \times X_0 \xrightarrow{(\gamma,\gamma)} X_0^d \times X_0^d.$$

Definition

A morphism $F : X \to Y$ in $\mathrm{Cat}^2_{\mathrm{wg}}$ is a 2-equivalence if

(i) For all $a, b \in X_0^d$ $F(a,b) : X(a,b) \to Y(Fa, Fb)$ is an equivalence of categories.

(ii) $p^{(1)}F$ is an equivalence of categories.

Coloured categories

- A coloured category is a category $\mathcal{C}$ with a subcategory $\mathcal{W}$ containing all objects. The arrows of $\mathcal{W}$ are called coloured arrows.
- Morphisms of colored categories are colour-preserving functors.
- A coloured graph is a graph in which some of the edges have been singled out as colours.
- To form the free coloured category on a coloured graph take the free category on the whole graph an let $\mathcal{W}$ be the free category on the coloured part of the graph.

Coloured ordinals

- A (finite) coloured ordinal is a free coloured category on a (finite) linearly ordered coloured graph.

- Let $\mathbb{T}$ be the category of finite non-empty coloured ordinals

  Morphisms are as usual ordinals for the dots but a link can be set but may not be broken.

- Functor $\pi : \mathbb{T} \to \Delta$ contracting all the links.

Semi-categories

- Let Δ_{mono} be obtained from Δ by removing the degeneracy maps.
- If $X \in [\Delta^{op}_{mono}, \mathrm{Set}]$ satisfies the Segal condition

$$X_k \cong X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \qquad k \geq 2$$

  then X is a semi-category.
- A coloured semi-category is a semi-category with a sub-semi-category comprising all objets. A morphism between coloured semi-categories is a colour preserving semi-functor.

The fat delta

Definition (J. Kock)

The fat delta $\underline{\Delta}$ is the category of all finite non-empty coloured semi-ordinals.

- One can naturally identify $\underline{\Delta} = \mathbb{T}_{mono}$.
- The functor $\pi : \mathbb{T} \to \Delta$ gives rise to a functor

$$\pi : \underline{\Delta} = \mathbb{T}_{mono} \to \Delta_{mono} \hookrightarrow \Delta \ .$$

Fair 2-categories

- Let Cat be the coloured category with coloured arrows the equivalences of categories.

Definition (J. Kock)

A fair 2-category is a colour-preserving functor $X : \underline{\Delta}^{op} \to \mathrm{Cat}$ preserving discrete objects and pullbacks over discrete objects.

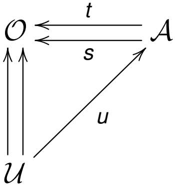
- Denote

$$\mathcal{O} = X_{\bullet}\,, \qquad \mathcal{A} = X_{\substack{\bullet \\ \bullet}}\,, \qquad \mathcal{U} = X_{\substack{\bullet \\ | \\ \bullet}}$$

and think of these as objects, arrows, weak identity arrows.

To give a fair 2-category X it is enough to give the following data:

a) A discrete category of objects $\mathcal{O} = X_{\bullet}$, a category of arrows $\mathcal{A} = X_{\substack{\bullet \\ \bullet}}$ and a category of weak units $\mathcal{U} = X_{\substack{\bullet \\ | \\ \bullet}}$ together with a commuting diagram



Fair 2-categories, cont.

b) Semi-category structures on $\mathcal{U} \rightrightarrows \mathcal{O}$ and $\mathcal{A} \rightrightarrows \mathcal{O}$ such that

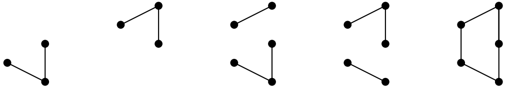
$$\begin{array}{ccc} \mathcal{U} & \rightrightarrows & \mathcal{O} \\ \downarrow & & \| \\ \mathcal{A} & \rightrightarrows & \mathcal{O} \end{array}$$

is a semi-functor.

c) The maps $\mathcal{U} \rightrightarrows \mathcal{O}$ as well as the composition maps

$$\mathcal{U} \times_{\mathcal{O}} \mathcal{A} \to \mathcal{A} \leftarrow \mathcal{A} \times_{\mathcal{O}} \mathcal{U}, \qquad \mathcal{U} \times_{\mathcal{O}} \mathcal{U} \to \mathcal{U}$$

which are images of



are equivalences of categories.

2-Equivalences in Fair^2

- There is a truncation functor

$$p^{(1)} : \mathsf{Fair}^2 \to \mathsf{Cat}$$

given by $(p^{(1)}X)_n = p(X_n)$ for all $n \in \Delta^{op}_{mono}$.

- Given $a, b \in X_0$ let $X(a, b)$ be the fiber at (a, b) of the map $X_1 \xrightarrow{(\partial_0,\partial_1)} X_0 \times X_0$.

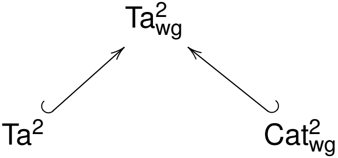
Definition

A morphism $f : X \to Y$ in Fair^2 is a 2-equivalence if

(i) For all $a, b \in X_0$, $f_{(a,b)} : X(a, b) \to Y(fa, fb)$ is an equivalence of categories.

(ii) $p^{(1)}f$ is an equivalence of categories.

Segal-type models



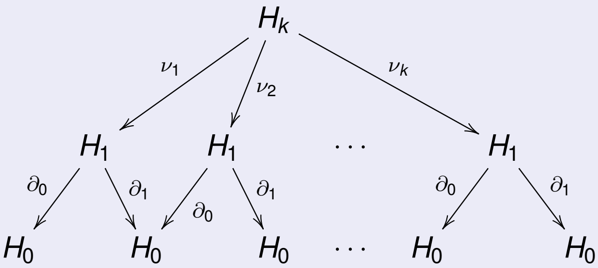
Definition

The category Ta^2_{wg} of weakly globular Tamsamani 2-categories is the full subcategory of $[\Delta^{op}, \text{Cat}]$ whose objects X are such that

i) X_0 is an equivalence relation.

ii) The induced Segal maps $\hat{\mu}_k : X_k \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1$ are equivalences of categories for all $k \geq 2$.

Let $H \in \mathrm{Ps}[\Delta^{op}, \mathrm{Cat}]$ be such that H_0 is discrete. The following diagram in Cat commutes



Hence there is a unique Segal map for all $k \geq 2$

$$H_k \to H_1 \times_{H_0} \overset{k}{\cdots} \times_{H_0} H_1 \ .$$

Segalic pseudo-functors

Definition

The category $\mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}\,]$ is the full subcategory of $\mathrm{Ps}[\Delta^{op}, \mathrm{Cat}\,]$ whose objects H are such that

i) H_0 is discrete.

ii) All Segal maps are isomorphisms for all $k \geq 2$

$$H_k \cong H_1 \times_{H_0} \overset{k}{\cdots} \times_{H_0} H_1 \ .$$

The functor Tr_2

Theorem

There is a functor

$$Tr_2 : \mathrm{Ta}^2_{\mathrm{wg}} \to \mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}]$$

$$(Tr_2 X)_k = \begin{cases} X_0^d, & k = 0 \\ X_1, & k = 1 \\ X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1, & k > 1\,. \end{cases}$$

Further, the strictification functor $St : \mathrm{Ps}[\Delta^{op}, \mathrm{Cat}] \to [\Delta^{op}, \mathrm{Cat}]$ *restricts to a functor*

$$St : \mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}] \to \mathrm{Cat}^2_{\mathrm{wg}}\,.$$

Strong Segalic pseudo-functors

- The inclusion functor $i : \Delta^{op}_{mono} \to \Delta^{op}$ induces a functor $i^* : \mathrm{Ps}[\Delta^{op}, \mathrm{Cat}\,] \to \mathrm{Ps}[\Delta^{op}_{mono}, \mathrm{Cat}\,]$.

Definition

A Segalic pseudo-functor $X \in \mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}\,]$ is called strong if $i^*X \in [\Delta^{op}_{mono}, \mathrm{Cat}\,]$. A morphism of strong Segalic pseudo-functors is a pseudo-natural transformation F in $\mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}\,]$ such that i^*F is a natural transformation in $[\Delta^{op}_{mono}, \mathrm{Cat}\,]$.

- We denote by $\mathrm{SSegPs}[\Delta^{op}, \mathrm{Cat}\,]$ the category of strong Segalic pseudo-functors, so that

$$i^* : \mathrm{SSegPs}[\Delta^{op}, \mathrm{Cat}\,] \to [\Delta^{op}_{mono}, \mathrm{Cat}\,]\ .$$

$\mathsf{Cat}^2_{\mathsf{wg}}$ and strong Segalic pseudo-functors

Proposition

The restriction to $\mathsf{Cat}^2_{\mathsf{wg}} \subset \mathsf{Ta}^2_{\mathsf{wg}}$ *of the functor* $Tr_2 : \mathsf{Ta}^2_{\mathsf{wg}} \to \mathsf{SegPs}[\Delta^{op}, \mathsf{Cat}]$ *is a functor*

$$Tr_2 : \mathsf{Cat}^2_{\mathsf{wg}} \to \mathsf{SSegPs}[\Delta^{op}, \mathsf{Cat}].$$

- To show that $i^* Tr_2 X \in [\Delta^{op}_{mono}, \mathsf{Cat}]$ we show that

$$\partial'_i = Tr_2 \partial_i : (Tr_2 X)_n \to (Tr_2 X)_{n-1}$$

satisfy the semi-simplicial identities $\partial'_i \partial'_j = \partial'_{j-1} \partial'_i, \quad i < j$.

Idea of proof

- The induced Segal maps ($k \geq 2$)

$$\hat{\mu}_k : X_k = X_1 \times_{X_0} \overset{k}{\cdots} \times_{X_0} X_1 \to X_1 \times_{X_0^d} \overset{k}{\cdots} \times_{X_0^d} X_1 = (Tr_2 X)_k$$

  is injective on objects, thus $\nu_k \hat{\mu}_k = \mathrm{Id}$, where ν_k is the pseudo-inverse.

- Thus for instance for $k > 2$

$$\begin{aligned}
&(Tr_2 X)_{k+1} \xrightarrow{\partial'_j} (Tr_2 X)_k \xrightarrow{\partial'_i} (Tr_2 X)_{k-1} \\
&\partial'_i \partial'_j = \hat{\mu}_{k-1} \partial_i \nu_k \hat{\mu}_k \partial_j \nu_{k+1} = \hat{\mu}_{k-1} \partial_i \partial_j \nu_{k+1} = \\
&= \hat{\mu}_{k-1} \partial_{j-1} \partial_i \nu_{k+1} = \hat{\mu}_{k-1} \partial_{j-1} \nu_k \hat{\mu}_k \partial_i \nu_{k+1} = \partial'_{j-1} \partial'_i \,.
\end{aligned}$$

Theorem

There is a functor

$$F_2 : \mathsf{Cat}^2_{\mathsf{wg}} \to \mathsf{Fair}^2$$

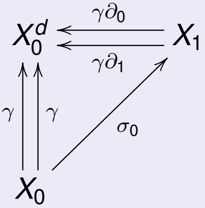
such that $(F_2 X)_0 = X^d_0$, $p^{(1)}X = p^{(1)}F_2 X$ *and, for each* $a, b \in X^d_0$, $X(a, b) \cong F_2 X(a, b)$.

F_2 *preserves* 2*-equivalences.*

- Given $X \in \mathsf{Cat}^2_{\mathrm{wg}}$ define

$$(F_2X)_{\bullet} = X_0^d, \qquad (F_1X)_{\substack{\bullet\\ \bullet}} = X_1, \qquad (F_2X)_{\substack{\bullet\\ |\\ \bullet}} = X_0$$

with the commuting diagram



where $\partial_0, \partial_1 : X_1 \to X_0$ (resp. $\sigma_0 : X_0 \to X_1$) are the face (resp. degeneracy) operators in X.

Idea of proof, cont.

- Since $i^* Tr_2 X \in [\Delta^{op}_{mono}, \mathrm{Cat}]$, $i^* Tr_2 X$ is a semi-category object internal to Cat,

$$X_1 \times_{X_0^d} X_0 \longrightarrow X_1 \underset{\gamma\partial_1}{\overset{\gamma\partial_0}{\rightrightarrows}} X_0^d \,.$$

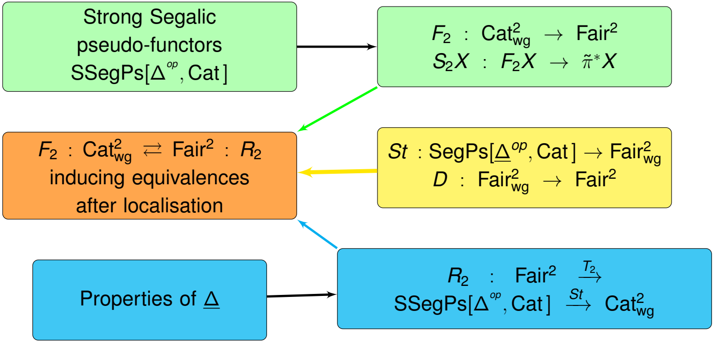
  which also restricts to a semi-category structure internal to Cat

$$X_0 \times_{X_0^d} X_0 \longrightarrow X_0 \underset{\gamma}{\overset{\gamma}{\rightrightarrows}} X_0^d \,.$$

- γ as well as the following composition maps are equivalences of categories

$$X_0 \times_{X_0^d} X_0 \to X_0, \qquad X_0 \times_{X_0^d} X_1 \to X_1, \qquad X_1 \times_{X_0^d} X_0 \to X_1$$

Overview



Plan:
- Background: the key players, $\mathrm{Cat}^2_{\mathrm{wg}}$, Fair^2, $\mathrm{SegPs}[\Delta^{op}, \mathrm{Cat}]$
- From $\mathrm{Cat}^2_{\mathrm{wg}}$ to Fair^2
- From Fair^2 to $\mathrm{Cat}^2_{\mathrm{wg}}$
- Sketch of proof of main result

Proposition

There is a functor

$$T_2 : \text{Fair}^2 \to \text{SSegPs}[\Delta^{op}, \text{Cat}]$$

such that, for each $X \in \text{Fair}^2$, $(T_2X)_0 = X_0$, $(T_2X)_1 = X_1$ *and* $(T_2X)_r = X_1 \times_{X_0} \overset{r}{\cdots} \times_{X_0} X_1$ *for* $r \geq 2$.

The functor T_2

- For each $\underline{k} \in \underline{\Delta}$ and $X \in \text{Fair}^2$ there is an equivalence of categories
$$\alpha_{\underline{k}} : X_{\pi(\underline{k})} \leftrightarrows X_{\underline{k}} : \beta_{\underline{k}}$$
such that $\beta_{\underline{k}}\alpha_{\underline{k}} = \text{Id}$.

- Let $\underline{f} : \underline{n} \to \underline{m}$ and $\underline{f}' : \underline{n}' \to \underline{m}'$ be maps in $\underline{\Delta}^{op}$ with $\pi \underline{f} = \pi \underline{f}'$. Then, if $X \in \text{Fair}^2$, $\beta_{\underline{m}} X(\underline{f})\alpha_{\underline{n}} = \beta_{\underline{m}'} X(\underline{f}')\alpha_{\underline{n}'}$.

- Given $X \in \text{Fair}^2$ and $n \in \Delta^{op}$, let $(T_2 X)_n = X_n$.

- Given $f : n \to m$ in Δ^{op}, choose $\underline{f} : \underline{n} \to \underline{m}$ in $\underline{\Delta}^{op}$ with $\pi \underline{f} = f$ and let $T_2 f$ be given by the composite

$$X_n \xrightarrow{\alpha_{\underline{n}}} X_{\underline{n}} \xrightarrow{\underline{f}} X_{\underline{m}} \xrightarrow{\beta_{\underline{m}}} X_m \ .$$

- From the previous slide, this is well defined.

- Given $n \xrightarrow{f} m \xrightarrow{g} s$ in Δ^{op}, to define $T_2(gf)$ we need maps in $\underline{\Delta}^{op}$

$$\underline{n} \xrightarrow{\underline{f}} \underline{m} \xrightarrow{\underline{g}} s, \pi(\underline{f}) = f, \pi(\underline{g}) = g,$$

  so that $T_2(gf)$ is the composite $X_n \xrightarrow{\alpha_{\underline{n}}} X_{\underline{n}} \xrightarrow{\underline{gf}} X_{\underline{s}} \xrightarrow{\beta_{\underline{m}}} X_m$.

- The existence of the liftings $\underline{f}, \underline{g}$ of f and g is not obvious.

- Main issue: one can easily find maps

$$\underline{n} \xrightarrow{\underline{f}'} \underline{m} \qquad \underline{m}' \xrightarrow{\underline{g}'} \underline{s} \qquad \pi(\underline{f}') = f, \; \pi(\underline{g}') = g$$

  but why can we ensure that we can find maps such that $\underline{m} = \underline{m}'$?

Lifting sequences of maps in delta

Proposition

Given maps in Δ

$$n_1 \xrightarrow{f_1} n_2 \xrightarrow{f_2} n_3 \rightarrow \cdots \xrightarrow{f_k} n_{k+1}$$

there are maps in $\underline{\Delta}$

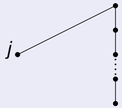
$$\underline{n}_1 \xrightarrow{\underline{f}_1} \underline{n}_2 \xrightarrow{\underline{f}_2} \underline{n}_3 \rightarrow \cdots \xrightarrow{\underline{f}_k} \underline{n}_{k+1}$$

with $\pi \underline{f}_j = f_j$.

- The proof is by induction on k and depends on properties of $\underline{\Delta}$ in relation to Δ.

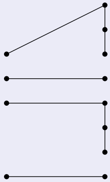
The map $\nu_{\underline{n}}$

- For each $n \in \Delta$ and $\underline{n} \in \underline{\Delta}$ such that $\pi(\underline{n}) = n$ let $\nu_{\underline{n}} : n \to \underline{n}$ be the map in $\underline{\Delta}$ which sends



where the link on the right contracts to j under π.

- For instance, the map $\nu_{\underline{4}} : 4 \to \underline{4}$ is



Some pushouts in fat delta

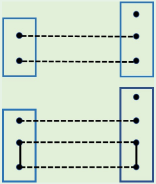
Lemma

Consider the diagram in $\underline{\Delta}$

$$\begin{array}{ccc} n & \xrightarrow{\varepsilon} & m \\ {\scriptstyle \nu_{\underline{n}}}\downarrow & & \\ \underline{n} & & \end{array}$$

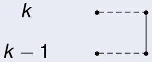
where $\nu_{\underline{n}}$ *is as in the previous slide and* $\varepsilon \in \Delta_{mono}$. *The pushout* $\underline{p}$ *of this diagram exists in* $\underline{\Delta}$ *and* $\pi(\underline{p}) = m$.

Example

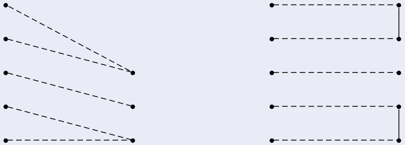


Lifting epimorphisms

- Let $\eta : n \to s$ be an epimorphism in Δ. There exists a map $\underline{\eta} : n \to \underline{s}$ in $\underline{\Delta}$ with $\pi(\underline{\eta}) = \eta$. Let $\underline{s}$ be obtained from s by inserting a link at each element j $(1 \leq j \leq s)$ such that $\eta(k) = \eta(k-1) = j$ for some $1 \leq k \leq n$. Then $\underline{\eta}$ sends



- For instance:



$\eta : 4 \to 2$ $\quad\quad$ $\underline{\eta} : 4 \to \underline{2}$

Proof of proposition, case $k = 1$

- Given $n_1 \xrightarrow{f_1} n_2$ in Δ, take the epi-mono factorization

$$n_1 \xrightarrow{\eta_1} s_1 \xrightarrow{\varepsilon_1} n_2$$

and build lifts in $\underline{\Delta}$

$$\begin{array}{ccccc} n_1 & \overset{\eta_1}{\dashrightarrow} & s_1 & \xrightarrow{\varepsilon_1} & n_2 \\ & \searrow & \downarrow \nu_{\underline{s}_1} & & \downarrow \\ & & \underline{s}_1 & \longrightarrow & \underline{n}_2 \end{array}$$

where $\underline{n}_2$ is the pushout.

Proof of proposition: inductive step

- Inductive hypothesis: we can lift $n_1 \xrightarrow{f_{k-1}\cdots f_2 f_1} n_k$ to

$$\underline{n}_1 \xrightarrow{\underline{f}_{k-1}\cdots \underline{f}_1} \underline{n}_k.$$

- Take the epi-mono factorization of $f_k : n_k \to n_{k+1}$

$$n_k \xrightarrow{\eta_k} s_k \xrightarrow{\varepsilon_k} n_{k+1}.$$

- Main issue: we know how to lift η_k to $\underline{\eta}_k : n_k \to \underline{s}_k$ but we need a lift that starts from $\underline{n}_k$.

Lifting epimorphisms, general case

Lemma

Given an epimorphism $\eta : n \to m$ *in* Δ *and given* $\underline{n} \in \underline{\Delta}$ *with* $\pi(\underline{n}) = n$, *there are maps in* $\underline{\Delta}$

$$\begin{array}{ccc} n & & m \\ \downarrow & & \downarrow \\ \underline{n} & \xrightarrow[\underline{f}]{} & \underline{m} \end{array}$$

with $\pi(\underline{f}) = \eta$.

- Applying the above lemma to η_k and $\underline{n}_k$ we build maps m in $\underline{\Delta}$

$$
\begin{array}{ccccc}
n_k & & s_k & \xrightarrow{\varepsilon_k} & n_{k+1} \\
\downarrow & & \downarrow{\scriptstyle \nu_{\underline{s}_k}} & & \downarrow \\
\underline{n}_k & \xrightarrow[\underline{\eta}_k]{} & \underline{s}_k & \xrightarrow[\underline{\varepsilon}_k]{} & \underline{n}_{k+1}
\end{array}
$$

  where $\underline{n}_{k+1}$ is the pushout.

- Thus $\underline{\varepsilon}_k \underline{\eta}_k$ is the required lift of $\varepsilon_k \eta_k = f_k$ completing the inductive step.

From Fair^2 to $\mathsf{Cat}^2_{\mathsf{wg}}$, cont.

Definition

Let $R_2 : \mathsf{Fair}^2 \to \mathsf{Cat}^2_{\mathsf{wg}}$ be the composite

$$\mathsf{Fair}^2 \xrightarrow{T_2} \mathsf{SSegPs}[\Delta^{op}, \mathsf{Cat}] \xrightarrow{St} \mathsf{Cat}^2_{\mathsf{wg}} \ ,$$

where St is the restriction to $\mathsf{SSegPs}[\Delta^{op}, \mathsf{Cat}]$ of the functor $St : \mathsf{SegPs}[\Delta^{op}, \mathsf{Cat}] \to \mathsf{Cat}^2_{\mathsf{wg}}$.

Comparison result

Theorem (P.)

The functors

$$F_2 : \mathrm{Cat}^2_{\mathrm{wg}} \rightleftarrows \mathrm{Fair}^2 : R_2$$

induce an equivalence of categories after localization with respect to the 2-equivalences

$$\mathrm{Cat}^2_{\mathrm{wg}}/\sim \;\simeq\; \mathrm{Fair}^2/\sim \,.$$

Method of proof

- Given $X \in \mathrm{Cat}^2_{\mathrm{wg}}$, we produce a 2-equivalence in $\mathrm{Cat}^2_{\mathrm{wg}}$ between X and $R_2 F_2 X$.
- Given $Y \in \mathrm{Fair}^2$, we produce a zig-zag of 2-equivalences in Fair^2 between Y and $F_2 R_2 Y$.
- The construction of these maps requires a new player, the category $\mathrm{Fair}^2_{\mathrm{wg}}$ of weakly globular fair 2-categories.

Proof of main result: comparing X and R_2F_2X

- Recall $R_2 : \mathrm{Fair}^2 \xrightarrow{T_2} \mathrm{SSegPs}[\Delta^{op}, \mathrm{Cat}\,] \xrightarrow{St} \mathrm{Cat}^2_{\mathrm{wg}}$ and $F_2 : \mathrm{Cat}^2_{\mathrm{wg}} \to \mathrm{Fair}^2$.
- Given $X \in \mathrm{Cat}^2_{\mathrm{wg}}$ there is a levelwise equivalence pseudo-natural transformation in $T_2F_2X \to X$ in $\mathrm{Ps}[\Delta^{op}, \mathrm{Cat}\,]$.
- By adjunction, this corresponds to a levelwise equivalence natural transformation in $[\Delta^{op}, \mathrm{Cat}\,]$

$$R_2F_2X = St\, T_2F_2X \to X\ .$$

- In particular, this is a 2-equivalence in Fair^2 between X and R_2F_2X. Hence $X \cong R_2F_2X$ in $\mathrm{Cat}^2_{\mathrm{wg}}/\sim$.

Proof of main result: comparing Y and $R_2 F_2 Y$

- Given $Y \in \text{Fair}^2$, there is a levelwise equivalence pseudo-natural transformation $F_2 R_2 Y \to Y$ in $\text{Ps}[\underline{\Delta}^{op}, \text{Cat}]$.

- By adjunction, this gives a natural transformation in $[\underline{\Delta}^{op}, \text{Cat}]$ $St\, F_2 R_2 Y \to Y$.

- Since $F_2 St\, T_2 Y \in \text{Fair}^2$ then $F_2 R_2 Y \in \text{SegPs}[\underline{\Delta}^{op}, \text{Cat}]$ so $St\, F_2 R_2 Y \in \text{Fair}^2_{\text{wg}}$.

- So we have a zig-zag of 2-equivalences in $\text{Fair}^2_{\text{wg}}$

$$F_2 R_2 Y \leftarrow St\, F_2 R_2 Y \to Y$$

Comparing Y and $R_2 F_2 Y$, cont.

- There is a functor $D : \mathrm{Fair}^2_{\mathrm{wg}} \to \mathrm{Fair}^2$ which preseves 2-equivalences and is identity on Fair^2.

- From the zig-zag of 2-equivalences in $\mathrm{Fair}^2_{\mathrm{wg}}$

$$F_2 R_2 Y \leftarrow St\, F_2 R_2 Y \to Y$$

  we obtain the zig-zag of 2-equivalences in Fair^2

$$F_2 R_2 Y = DF_2 R_2 Y \leftarrow DSt\, F_2 R_2 Y \to DY = Y\ .$$

- It follows that $Y \cong R_2 F_2 Y$ in $\mathrm{Fair}^2/\sim$.

Summary

- Several models of weak 2-categories, in particular the Segal-type models and fair 2-categories.
- Direct comparison between weakly globular double categories and fair 2-categories.
- New light on weakly globular double categories, as encoding weak units.
- Lifting of strings of maps from Δ to $\underline{\Delta}$; category $\mathsf{Fair}^2_{\mathsf{wg}}$.
- Potential for higher dimensional generalisations.