# Identities in higher categories (in dependent type theory)

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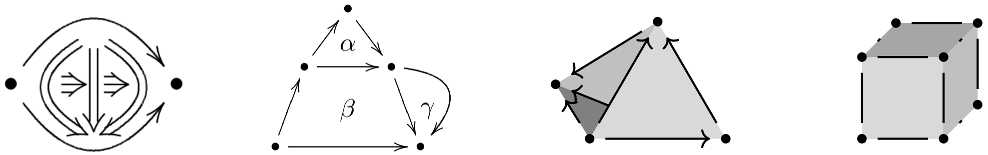
## General goal:

Develop a theory of $(\infty, 1)$-categories in homotopy type theory.

## Motivations:

1. These structures are already there (e.g. a universe $\mathcal{U}$).
2. Expected to be key to the question "Can HoTT eat itself?"
3. Useful for addressing other open problems, cf. Christian Sattler's talk ("Is the suspension of a set 1-truncated?")




## Approach:

I use the simplicial approach (*Segal spaces*); cf. Eric Finster's talk for an opetopic definition.

## Caveat:

We want a "semi-synthetic" (type = space) formulation of higher categories (not a set-based one).

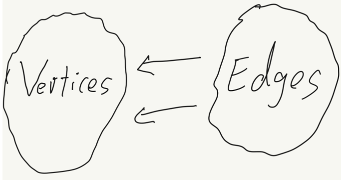
# PART 1

Why are higher-dimensional **semi-categories**
easier to define than higher-dimensional **categories**
in type theory?

(I.e.: What makes identities difficult?)

Structures can often be defined as presheaves over some category (plus properties).

Example: Directed graphs are presheaves on the category $\bullet \rightrightarrows \bullet$




Definition of a graph in type theory:

$$V : \mathcal{U} \qquad V' : \mathcal{U}$$
$$E : \mathcal{U} \qquad E' : V' \times V' \to \mathcal{U}$$
$$s : E \to V$$
$$t : E \to V$$

The two definitions are equivalent (as *records* or *nested* $\Sigma$ *types*).

$(V, E, s, t) \mapsto (V'E')$ with $V' :\equiv V$ and $E'(a, b) :\equiv \Sigma(v : V).(s(v) = a) \times (t(v) = b)$

$(V', E') \mapsto (V, E, s, t)$ with $V' :\equiv V$ and $E'(a, b) :\equiv \Sigma(v : V).(s(v) = a) \times (t(v) = b)$

Continued example: Directed graphs as presheaves on the category $\bullet \rightrightarrows \bullet$

$$V : \mathcal{U} \qquad E : \mathcal{U} \qquad s : E \to V \qquad t : E \to V$$

"Tedious definition"

$$V' : \mathcal{U} \qquad E' : V' \times V' \to \mathcal{U}$$

"Economical definition"

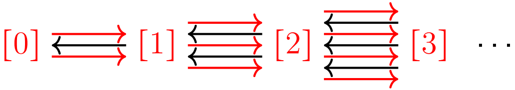
**Caveat:**

- $\mathcal{U}$ is a 1-category with categorical laws are given by judgmental equality.
- $\mathcal{U}$ is a higher category with higher cells given by the internal equality type.

The first is meta-theoretic, the second is internal.

$\Rightarrow$ It's a good idea to be economical!

# $(n, 1)$-categories as presheaves on $\Delta$?

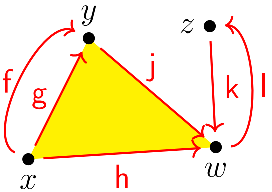



$A_0 : \mathcal{U}$

$A_1 : A_0 \to A_0 \to \mathcal{U}$

$A_2 : (x, y, z : A_0) \to A_1(x, y) \to A_1(y, z) \to A_1(x, z) \to \mathcal{U}$

$A_3 : (x, y, z, w : A_0) \to \ldots$




Example:

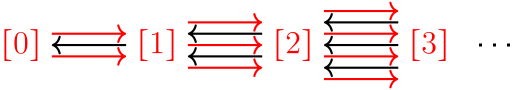
$A_0 \equiv \{x, y, z, w\}$

$A_1(x, y) \equiv \{f, g\}$

$A_1(x, w) \equiv \{h\}, \ldots$

$A_2(x, y, w, g, j, h) \equiv$ *yellow* $\Delta$

# $(n,1)$-categories as presheaves on $\Delta$?

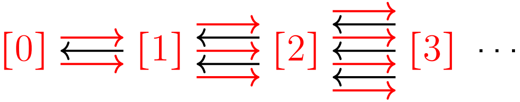



$$A_0 : \mathcal{U}$$
$$A_1 : A_0 \to A_0 \to \mathcal{U}$$
$$A_2 : (x, y, z : A_0) \to A_1(x, y) \to A_1(y, z) \to A_1(x, z) \to \mathcal{U}$$
$$A_3 : (x, y, z, w : A_0) \to \ldots$$

Note: The above represents the presheaf $\Delta_+^{\leq 2} \to \mathcal{U}$ given by

$$[0] \mapsto A_0$$
$$[1] \mapsto \Sigma(x, y : A_0), A_1(x, y)$$
$$[2] \mapsto \Sigma x, y, z, f, g, h, A_2(x, y, z, f, g, h)$$

# PART 2



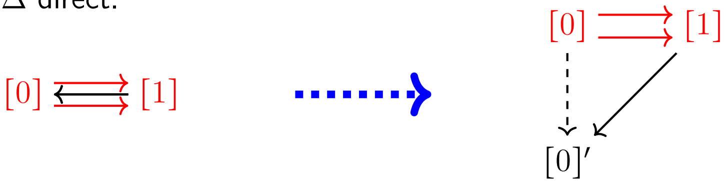

The "Reedy fibrant representation" (diagrams via type families) only tells us how to define a type of presheaves on the direct part $\Delta_+$.

How to add the inverse/negative part $\Delta_-$?

# Construction 1: A direct replacement construction

(Sattler's variation of Kock's *fat Delta*)

Idea: "Make $\Delta$ direct."




$A_0 : \mathcal{U}$

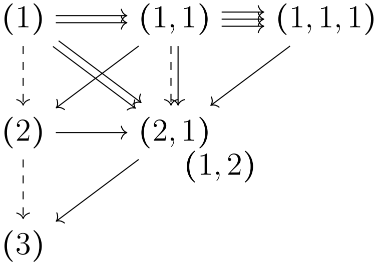
$A_1 : A_0 \to A_0 \to \mathcal{U}$

$A_{0'} : (x : A_0) \to A_1(x, x) \to \mathcal{U}$

$h : (x : A_0) \to \mathsf{isContractible}\,(\Sigma(i : A_1(x, x)).A_{0'}\, x\, i)$

The dashed/marked/thin morphism $[0] \to [0']$ gets mapped to an equivalence, expressed by $h$. Note: This is a proposition!

# Construction 1: A direct replacement construction




I now write $(1,1,1)$ instead of $[2]$, and so on.

Def. of this category:
Objects are non-empty lists of positive integers; morphisms from $(a_0, \ldots, a_m)$ to $(b_0, \ldots, b_n)$ are maps $f \in \Delta([m],[n])$ such that $b_j \geq$ the sum of all $f^{-1}[j]$.
$f$ is marked if it's an identity in $\Delta$.

In general: For $R$ a Reedy category, define the direct replacement $D(R)$ as follows:
Objects are arrows in $R_-$. A morphism between $s : x \to y$ and $t : z \to w$ is a morphism $f \in R(y, w)$ such that there exists a morphism $x \to w$ in $R_+$ that makes the square commute.

# Construction 2: Homotopy-coherent diagrams

Idea: "Make the tedious definition work."
I.e.: Drop the idea that we want to represent presheaves via type families.

Important example of a "semi-simplicial type": presheaf $\mathbf{T} : \Delta_+ \to \mathcal{U}$,

$$\begin{array}{lcl} \mathbf{T}_0 & \cong & \mathcal{U} \\ \mathbf{T}_1(X,Y) & \cong & X \to Y \\ \mathbf{T}_2(X,Y,Z,f,g,h) & \cong & g \circ f = h \\ \ldots & \ldots & \ldots \end{array}$$

(E.g. constructed as Reedy fibrant replacement of the semi-simplicial nerve of $\mathcal{U}$.
This is very roughly Shulman's universe with relations replaced by functions.)

# Construction 2: Homotopy-coherent diagrams

For $\mathcal{C}$ a category, write $N(\mathcal{C})$ for the nerve (chains of morphisms).

Define a *homotopy coherent presheaf* on $\mathcal{C}$ to be a "natural transformation" $N(\mathcal{C}^{\mathsf{op}}) \to \mathbf{T}$; formally:

**Definition: homotopy coherent diagram**

The type of homotopy coherent presheaves is the Reedy limit of the composition $\left(\int N(\mathcal{C}^{\mathsf{op}})\right) \xrightarrow{\mathsf{shape}} \Delta_+^{\mathsf{op}} \xrightarrow{\mathbf{T}} \mathsf{Type}$.

Intuition of such a "natural transformation":

- level 0: For every object $x$ of $\mathcal{C}$, a type $A_x : \mathcal{U}$;
- level 1: For every arrow $x \xrightarrow{f} y$ in $\mathcal{C}^{\mathsf{op}}$, a function $A_g : A_x \to A_y$;
- level 2: For every chain $x \xrightarrow{f} y \xrightarrow{g} z$ in $\mathcal{C}^{\mathsf{op}}$, an equality $A_g \circ A_f = A_{g \circ f}$;
- level 3: For every chain $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{h}$ in $\mathcal{C}^{\mathsf{op}}$, a higher equality; ...

# Construction 2: Homotopy-coherent diagrams

**Result 1**

The type of homotopy coherent presheaves on $\Delta$ and the type of Reedy fibrant presheaves on the Kock/Sattler "fat" $\Delta$ are equivalent (in a theory where they exist – still unknown for pure HoTT).

1. Presheaves on $\Delta$ defined
2. To do: add Segal condition
3. $\Rightarrow$ Definition of $(\infty, 1)$-categories

(Un)surprisingly, step 2 is completely unproblematic.
Segal condition: *The usual maps* $A_n \to A_1 \times_{A_0} A_1 \times_{A_0} \ldots \times_{A_0} A_1$ *are equivalences.*

Note: That's a proposition.

# Construction 3: Idempotent equivalences

Start with a semi-simplicial type with Segal condition – an "$(\infty, 1)$-semicategory".

The Segal condition gives a notion of composition:

$$\_ \circ \_ : A_1(y, z) \times A_1(x, y) \to A_1(x, z).$$

Define:

- $f : A_1(x, x)$ is *idempotent* if $f \circ f = f$ (i.e. if we have $A_2(f, f, f)$).
- $f : A_1(x, y)$ is an *equivalence* if both $(f \circ \_)$ and $(\_ \circ f)$ are equivalences of types

Then, for any $x : A_0$, the type

$$\Sigma(i : A_1(x, x)).\text{is-idempotent}(i) \times \text{is-equivalence}(i)$$

is a proposition.

# Construction 3: Idempotent equivalences

Thus, we can define:

**Definition: $(\infty, 1)$-category**

A *simple* $(\infty, 1)$*-category* is a semi-simplicial type satisfying the Segal condition and such that every object is equipped with an idempotent equivalence.

**Result 2 (caveat: not properly written up yet)**

This simple notion of $\infty$-category is equivalent to both the definition via homotopy-coherent presheaves and the one via a direct replacement.

# A weak version of the result

**Result 2' (weak version of Result 2)**

Let $A$ be an $(\infty, 1)$-semicategory.
If $A$ has an idempotent equivalence, then we can construct all the degeneracy maps $s_i : A_n \to A_{n+1}$ such that the equalities

$$
\begin{aligned}
d_i \circ s_j &\equiv s_{j-1} \circ d_i && \text{if } i < j \\
d_i \circ s_j &\equiv s_j \circ d_{i-1} && \text{if } i > j+1 \\
d_i \circ s_j &\equiv \mathrm{id} && \text{if } i = j \text{ or } i = j+1
\end{aligned}
$$

hold judgmentally.

# Sketch of Result 2'

Let $\alpha$ be an $n$-simplex. We need to construct an $(n+1)$-simplex $s_i(\alpha)$. We construct $s_i(\alpha)$ and $s_i(s_i(\alpha))$ simultaneously, by induction on $n$.
Assume $n = i = 2$ for simplicity (it works in essentially the same way on all levels), and assume $\alpha$ is given by the chain $x \xrightarrow{f} y \xrightarrow{g} z$. Consider the partial $4$-simplex with "spine" $x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{i} z \xrightarrow{i} z$ and where all faces that we have by induction are filled in. One can then check manually that three faces at level $3$ are missing and the single face on level $4$ is missing. But the missing faces at level $3$ have the same boundary, and the problem is equivalent to an "ordinary" horn-filling problem; as usual, this is a re-formulation of the Segal condition.