Università di Genova

ItaCa

A 2-categorical representation of deduction

j.w.w. Ivan Di Liberti

LI 2022

Greta Coraglia

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How do rules *really* work, syntactically?
What about constructors/connectives?

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An account of context, judgement, deduction

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$$\begin{array}{c} \mathbb{F} \\ \downarrow f \\ \mathrm{ctx} \end{array}$$

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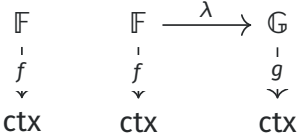

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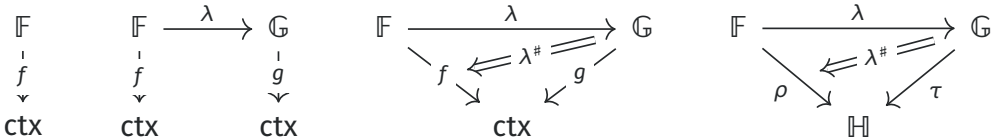
($\mathcal{P}$) *policies*, a class of 2-dimensional cells filling (some) triangles induced by rules (functors in $\mathcal{R}$) and judgements (functors in $\mathcal{J}$).



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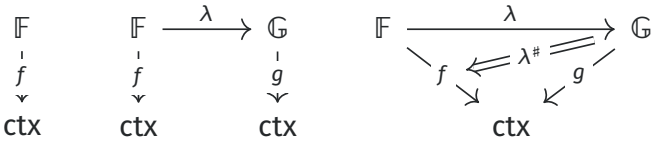

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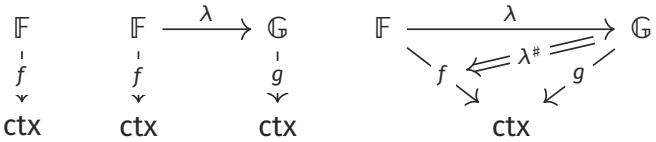

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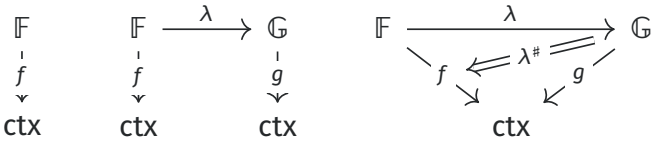

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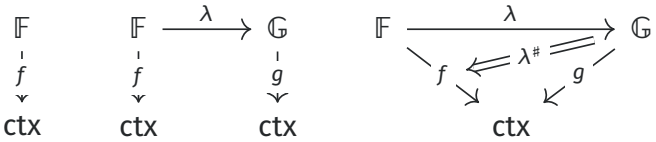

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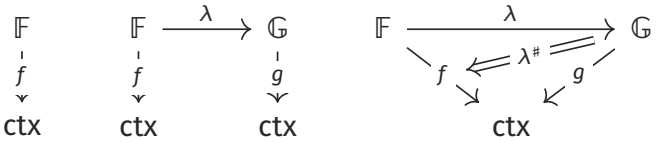

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and, possibly, Γ and $g\lambda F$ and related by a map

$$\lambda^{\#}_{F} : g\lambda F \to \Gamma$$

Example: toy MLTT

toy MLTT:

$$\begin{cases} \mathsf{ctx} = \text{contexts and substitutions} \\ \mathcal{J} = \{\dot{u}, u\} \\ \mathcal{R} = \{\Sigma\} \\ \mathcal{P} = \{id : u \circ \Sigma \Rightarrow \dot{u}\} \end{cases}$$

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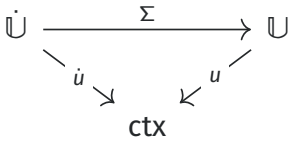

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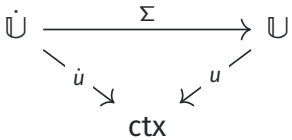

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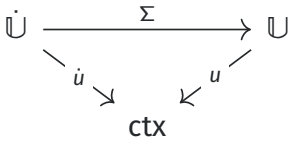

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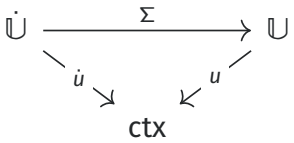

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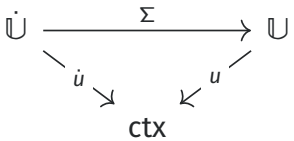

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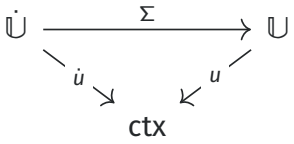

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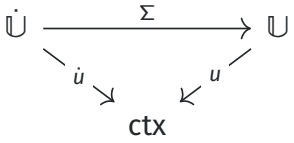

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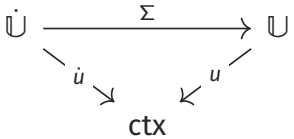

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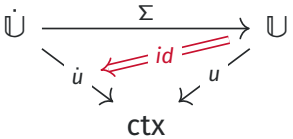

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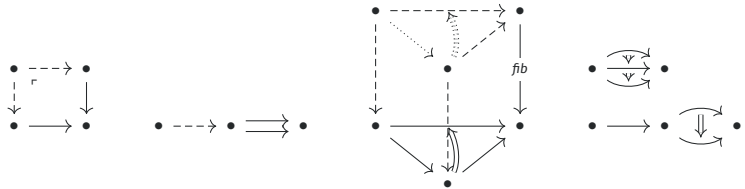

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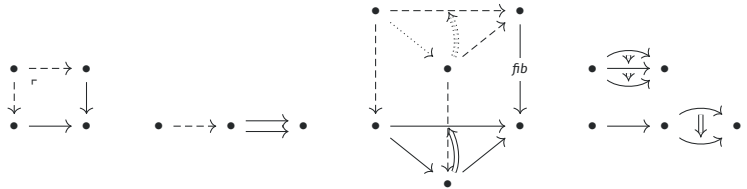

We now have a calculus!

Nested judgements

Pullbacks compute *nested judgements* such as

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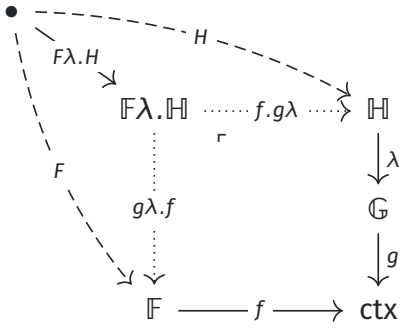

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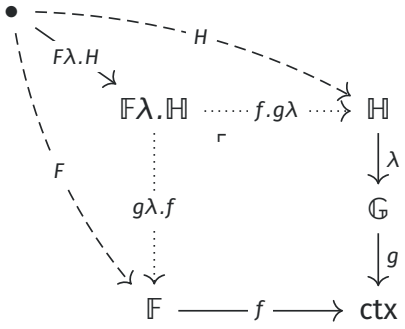

$$\Gamma \vdash F\lambda.H\ \mathbb{F}\lambda.\mathbb{H}$$

Nested judgements

Pullbacks compute *nested judgements* such as

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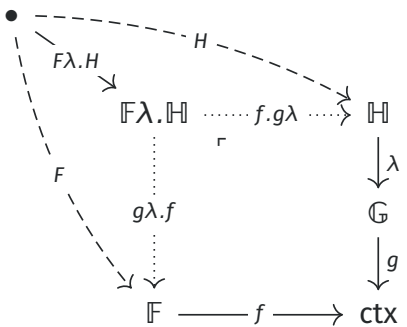

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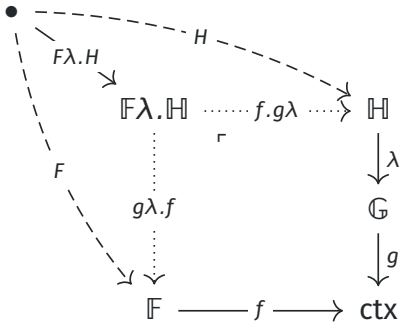

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Example: toy MLTT

toy MLTT: ctx, $\mathcal{J} = \{\dot{u}, u\}$, $\mathcal{R} = \{\Sigma\}$, $\mathcal{P} = \{id : u \circ \Sigma \Rightarrow \dot{u}\}$

In the judgemental theory generated by (ctx, $\mathcal{J}, \mathcal{R}, \mathcal{P}$) we find the following:

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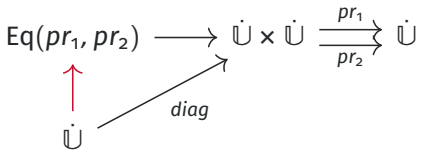

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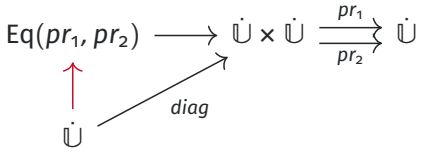

reads as $(\rho)\ \dfrac{\Gamma \vdash (a, A)\ \dot{\mathbb{U}}}{\Gamma \vdash \rho(a, A)\ \mathrm{Eq}(pr_1, pr_2)}$

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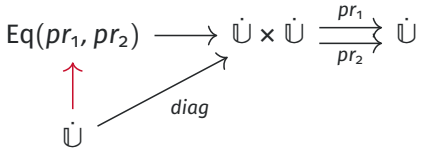

reads as

$$(\rho)\ \frac{\Gamma \vdash a : A}{\Gamma \vdash a =_A a}$$

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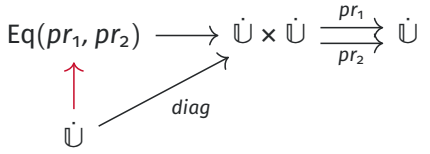

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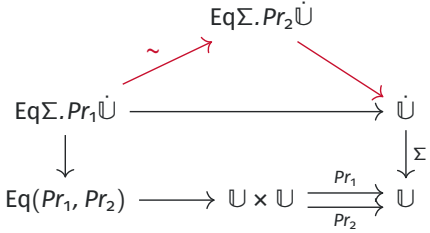

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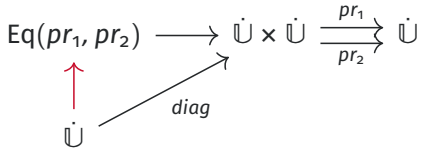

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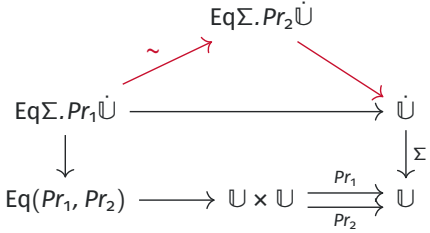

reads as $(\sigma)\ \dfrac{\Gamma \vdash (a, (A, B))\ \mathrm{Eq}\Sigma.Pr_1\dot{\mathbb{U}}}{\Gamma \vdash \sigma(a, (A, B))\ \dot{\mathbb{U}}}$

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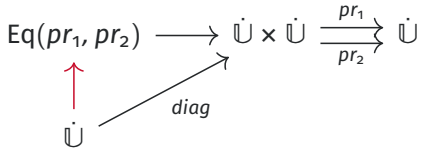

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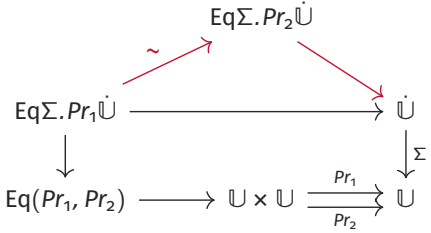

reads as $(\sigma)\ \dfrac{\Gamma \vdash a : A \quad \Gamma \vdash A = B}{\Gamma \vdash a : B}$

jDTT, I: definition

jDTT:

$$\begin{cases} \mathrm{ctx} = \text{contexts and substitutions} \\ \mathcal{J} = \{\dot{u}, u\}, \text{ with } u, \dot{u} \text{ fibrations} \\ \mathcal{R} = \{\Sigma, \Delta\}, \text{ with } \Sigma \dashv \Delta \\ \mathcal{P} = \{id : u \circ \Sigma \Rightarrow \dot{u}, \epsilon, \eta\}, \text{ with } \epsilon, \eta \text{ cartesian} \end{cases}$$

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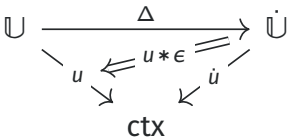

$$\frac{\Gamma \vdash A\ \mathbb{U}}{\dot{u}\Delta A \vdash \Delta A\ \dot{\mathbb{U}}}$$

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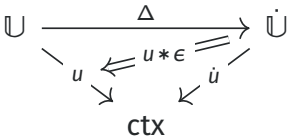

$$\frac{\Gamma \vdash A\ \mathbb{U}}{\dot{u}\Delta A \vdash \Delta A\ \dot{\mathbb{U}}} \qquad \frac{\Gamma \vdash A}{\Gamma.A \vdash x_A : A\delta_A}$$

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Theorem (1)

If $\dot{u}, u$ are discrete, the jDTT is (equivalent to) a natural model* à la Awodey.

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Theorem (1)

If $\dot{u}$, u are discrete, the jDTT is (equivalent to) a natural model* à la Awodey.

Theorem (2)

The judgmental theory generated by jDTT contains codes for all structural rules of dependent type theory.

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jDTT, II: coding dependent families

$$\frac{\Gamma \vdash a : A \quad \Gamma.A \vdash b : B}{\Gamma \vdash A \quad \Gamma.A \vdash b : B}$$

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jDTT, II: coding dependent families

$$\Gamma \vdash a : A \quad \Gamma.A \vdash b : B$$
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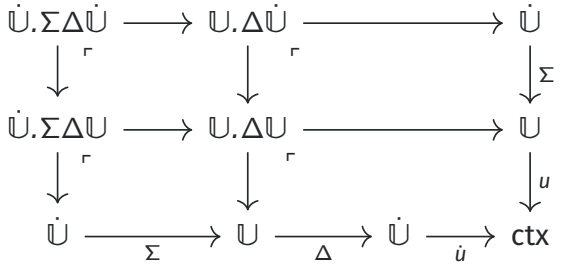

jDTT, II: coding dependent families

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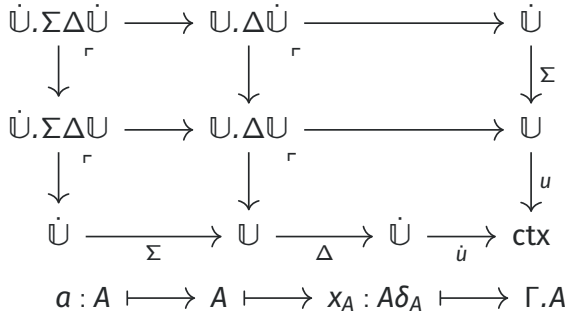

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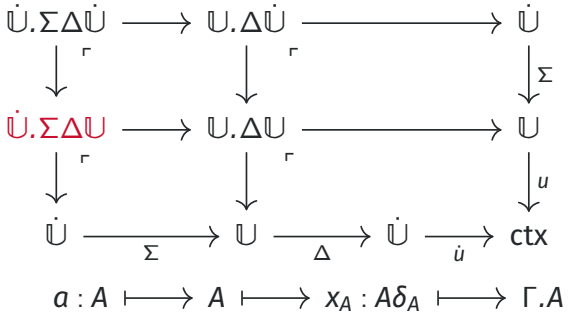

jDTT, III: policies for type dependency

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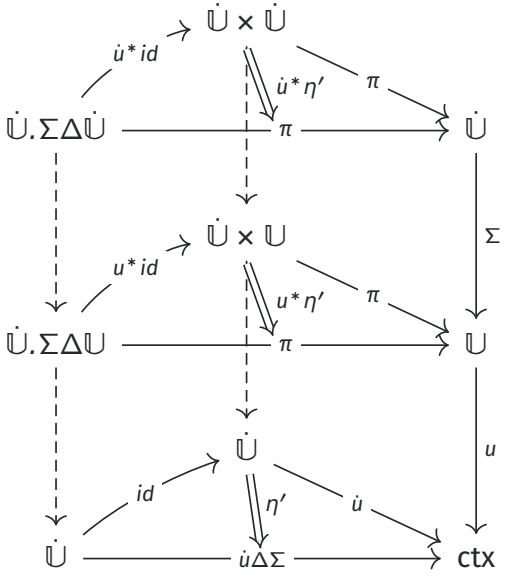

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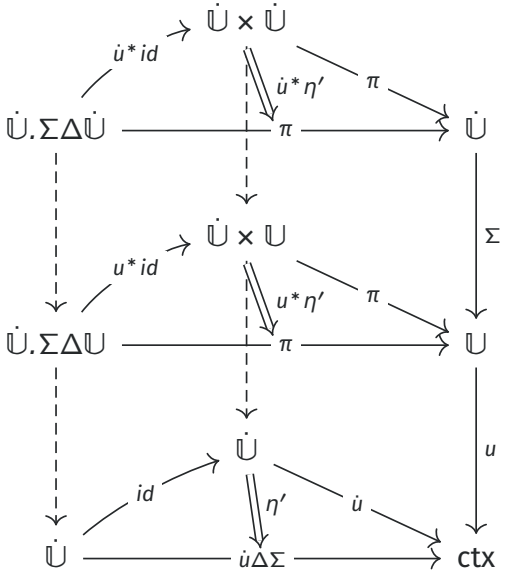
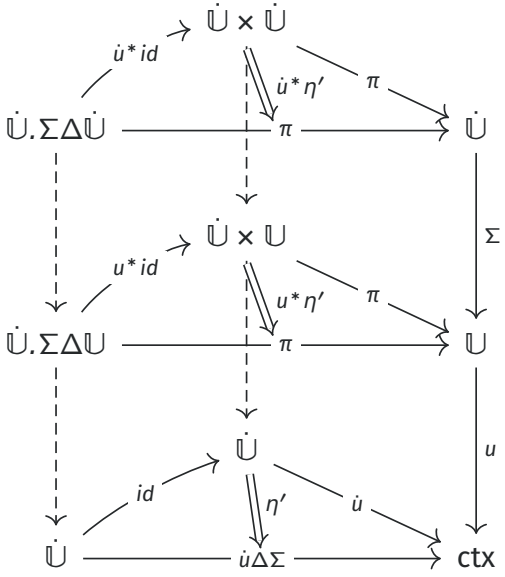


$\Gamma \longrightarrow \Gamma.A$

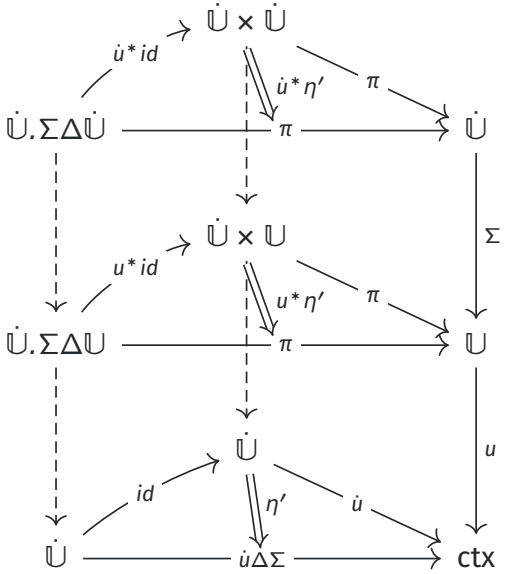
jDTT, III: policies for type dependency



$$(\pi\ u^*\ id)\ \frac{\Gamma.A \vdash (a, B)\ \dot{\mathcal{U}}.\Sigma\Delta\mathcal{U}}{\Gamma \vdash ??\ \mathcal{U}}$$

$$\Gamma \to \Gamma.A$$

jDTT, III: policies for type dependency



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$?? \to B$

$\Gamma \to \Gamma.A$

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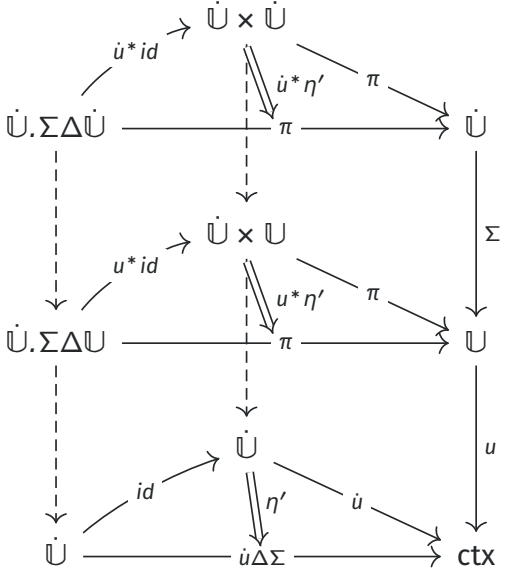

$$(\pi\ \dot{u}^*\ id)\ \frac{\Gamma.A \vdash (a, b)\ \dot{\mathbb{U}}.\Sigma\Delta\dot{\mathbb{U}}}{\Gamma \vdash ??\ \dot{\mathbb{U}}}\quad ?? \to b$$

$$(\pi\ u^*\ id)\ \frac{\Gamma.A \vdash (a, B)\ \dot{\mathbb{U}}.\Sigma\Delta\mathbb{U}}{\Gamma \vdash ??\ \mathbb{U}}\quad ?? \to B$$

$$\Gamma \to \Gamma.A$$

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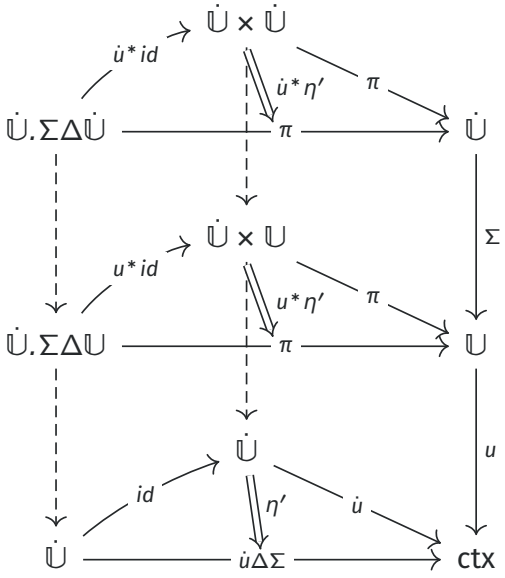

(Sbst') $\dfrac{\Gamma \vdash a : A \qquad \Gamma.A \vdash b : B}{\Gamma \vdash b[a] : B[a]}$

$b[a] \to b$

(Sbst) $\dfrac{\Gamma \vdash a : A \qquad \Gamma.A \vdash B}{\Gamma \vdash B[a]}$

$B[a] \to B$

$\Gamma \to \Gamma.A$

jDTT, IV: type constructors

Plus, we can define what diagrams one needs to add to jDTT in order to get type constructors.

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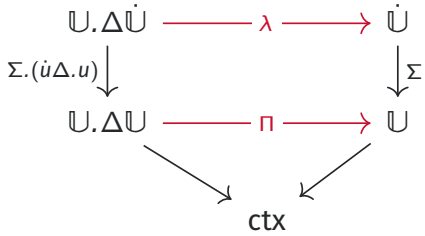
Theorem (3)

It has Π-*types* if it has two additional rules Π, λ such that the diagram below is commutative and the upper square is a pullback.



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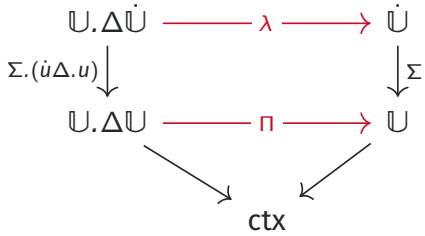

$$(\Pi F)\ \frac{\Gamma \vdash A \qquad \Gamma.A \vdash B}{\Gamma \vdash \Pi_A B}$$

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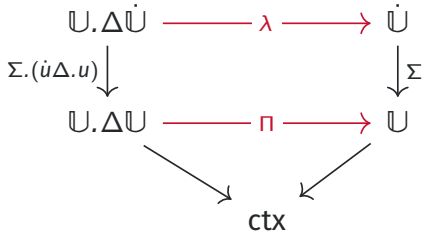

$$(\Pi\mathrm{I})\ \frac{\Gamma \vdash A \qquad \Gamma.A \vdash b : B}{\Gamma \vdash \lambda_A b : \Pi_A B}$$

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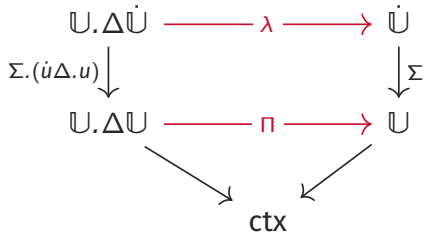

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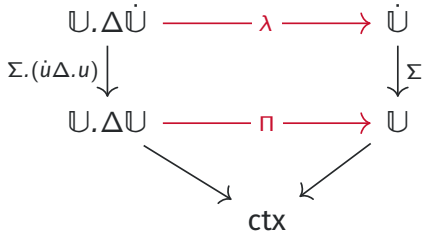

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(ΠE): the unique map induced by the pullback from the classifier of (A, B, f, a)
(ΠCη) and (ΠCβ): induced by the canonical isomorphism

jDTT, IV: type constructors

This generalizes quite nicely.

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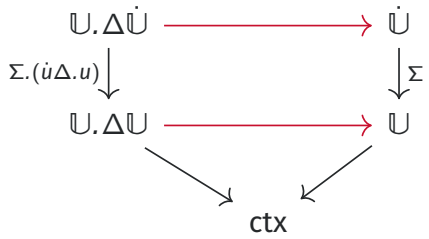
Pick

$$\Gamma \vdash A \quad \Gamma.A \vdash b : B$$

$$\Gamma \vdash A \quad \Gamma.A \vdash B$$



... get Π-types.

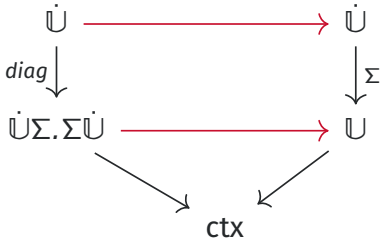
jDTT, IV: type constructors

This generalizes quite nicely.

Pick

$$\Gamma \vdash a : A$$

$$\Gamma \vdash a : A \quad \Gamma \vdash a' : A$$



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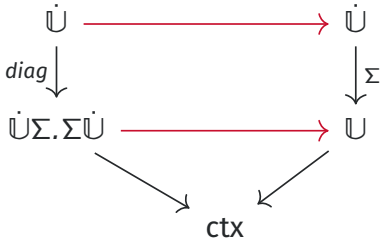
Pick

$$\Gamma \vdash a : A$$

$$\Gamma \vdash a : A \quad \Gamma \vdash a' : A$$



... get Id-types.

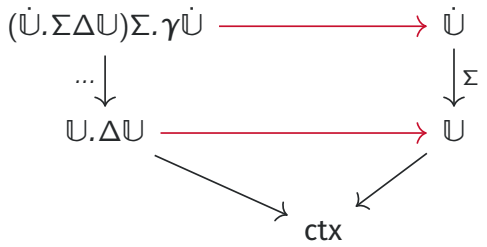
jDTT, IV: type constructors

This generalizes quite nicely.

Pick

$$\Gamma \vdash A \quad \Gamma.A \vdash B \quad \Gamma \vdash a : A \quad \Gamma \vdash b : B[a]$$

$$\Gamma \vdash A \quad \Gamma.A \vdash B$$



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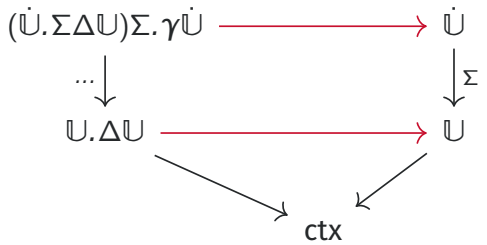

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$$(\vdash \Gamma \; ctx)$$

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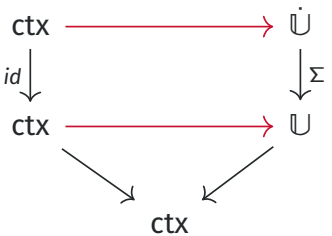

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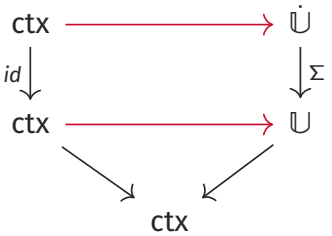
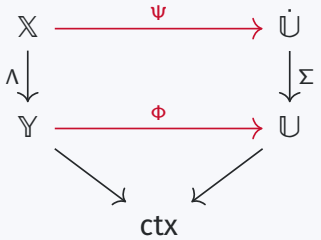


... get unit-types.

jDTT, IV: type constructors

General type constructor

A judgemental dependent type theory *with* Φ*-types* is a jDTT having two additional rules Φ, Ψ such that the diagram below is commutative and the upper square is a pullback.



And we can do calculations once for all of the above.

$$\text{(Sbst)}\ \frac{\Gamma \vdash a : A \qquad \Gamma.A \vdash B}{\Gamma \vdash B[a]}$$

$$\text{(Cut)}\ \frac{x;\Gamma \vdash \phi \qquad x;\Gamma, \phi \vdash \psi}{x;\Gamma \vdash \psi}$$

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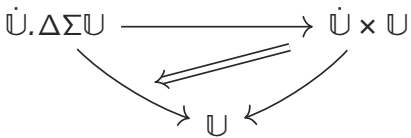

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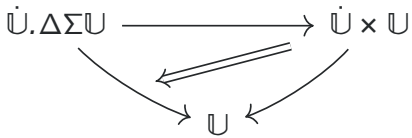

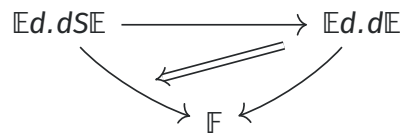

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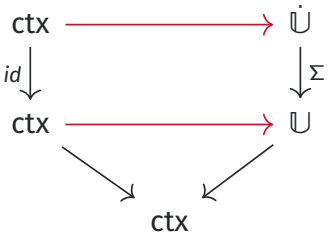

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Thank you for listening!