Abstract Strategies and Formal Coherence

Logic in Higher Structures

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Introduction

Overview and context

- Calculation of **cofibrant replacements** using higher-dimensional rewriting techniques.
  - Polygraphic resolutions: Métayer 2003, Guiraud-Malbos 2012, ...
- **Goal**: Formalisation of this theory in a proof assistant.
  - In **Isabelle**: C.-Malbos-Struth 2022 (WIP).
- **Method**: Obtain a general algebraic setting for **coherence proofs** capturing:
  - Polygraphic structure, in particular higher dimensions.
  - Proof mechanisms: Struth 2001, Desharnais-Möller-Struth 2004, ...
- **Today**: treat the case of 2-dimensional coherence proofs for abstract rewriting systems.

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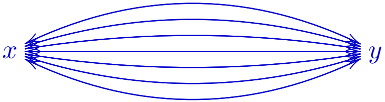

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- An ARS is **coherent** when any two parallel zig-zags are equivalent modulo some 'higher' relations.
  - Rewriting provides a constructive method for proving coherence,
  - and calculating a **generating set** of such relations.

Categorical framework for coherent abstract rewriting

- A **1-polygraph** ϕ is a directed graph, *i.e.* $\phi_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \phi_1$.

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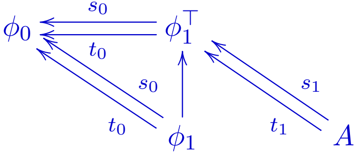

satisfying the globular relations

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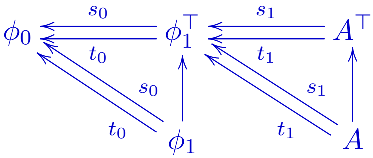

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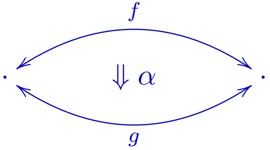
- The pair (ϕ, A) generates the **free 2-groupoid** $\phi^\top(A)$.

Coherence via convergent rewriting

- Given a 1-polygraph ϕ, a **homotopy basis** is a cellular extension A such that

$$\forall f, g \in \phi_1^{\top}$$
$$\exists \alpha \in \phi^{\top}(A), \alpha : f \Rightarrow g$$



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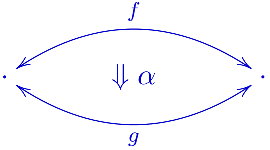

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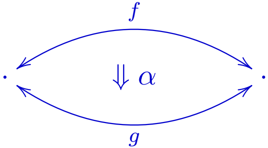

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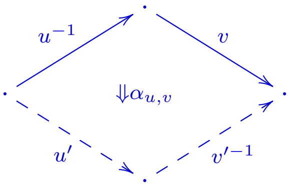
- When ϕ is confluent and terminating, *i.e.* **convergent**, a coherent extension of ϕ can be calculated using rewriting techniques.
- Indeed, the cellular extension consisting of a 2-cell $\alpha_{u,v}$ for every **local branching** (u, v) is a homotopy basis for ϕ.

$\forall u, v \in \phi_1, s_0(u) = s_0(v),$
$\exists u', v' \in \phi_1^*$
let $\alpha_{u,v} : u^{-1}v \Rightarrow u'v'^{-1}$



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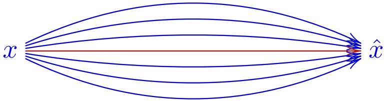

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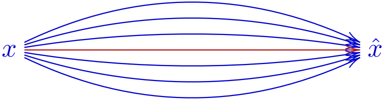

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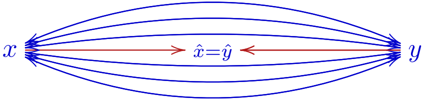

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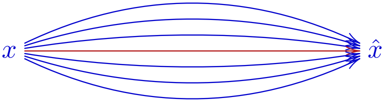

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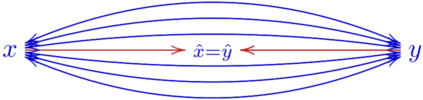
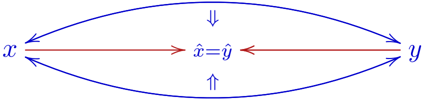


  - Proving coherence can be achieved by paving toward this canonical zig-zag:



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- **Goal**: describe...
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- **Abstract rewriting systems** and their properties have been formalised in **modal Kleene algebras (MKA)**.
- We introduce **globular 2-Kleene algebras**, extending MKA, as a natural setting for abstract coherence proofs.

Theorem (Abstract coherence theorem)

Let K be a Boolean globular 2-Kleene algebra (satisfying additional hypotheses) and $\phi \in K_1$ convergent. For any skeleton σ of $exh(\phi)$,

$$\overline{\phi} \odot_0 \phi \leq |A\rangle_1(\phi^{*_0} \odot_0 \overline{\phi}^{*_0}) \qquad \Rightarrow \qquad \phi^{\top_0} = (\phi + \overline{\phi})^{*_0} \leq |\hat{A}^{*_1}\rangle_1(\sigma \odot_0 \overline{\sigma}).$$

Rewriting in modal Kleene algebras

Modal Kleene algebras

- A **dioid** is a tuple $K = (K,\ +,\ 0,\ \cdot\ ,\ 1)$ such that
  - $(K, +, 0)$ is a commutative monoid.
  - $(K, \cdot, 1)$ is a monoid.
  - Distributivity: $(x + y) \cdot z = x \cdot z + x \cdot y$ and $x \cdot (y + z) = x \cdot y + x \cdot z$.
  - Annhilator: $x \cdot 0 = 0 = 0 \cdot x$
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  - Idempotence: $x + x = x$
  - We equip K with a an **order** $\leq$ given by

$$x \leq y \qquad \Longleftrightarrow \qquad x + y = y$$

Modal Kleene algebras

- A **Kleene algebra** is a dioid K equipped with a map

$$(-)^* : \; K \to K$$

satisfying the following axioms for all $x, y, z \in K$:

$$1 + xx^* \leq x^*, \qquad y + xz \leq z \Rightarrow x^*y \leq z,$$
$$1 + x^*x \leq x^*, \qquad y + zx \leq z \Rightarrow yx^* \leq z.$$

Modal Kleene algebras

- A Kleene algebra K along with a map $ad : K \to K$ satisfying:

$$ad(x)x = 0, \quad ad(xy) \leq ad(x\, ad^2(y)), \quad ad^2(x) + ad(x) = 1,$$

for all $x, y \in K$ is called a **(Boolean) Kleene algebra with domain**. The map $d := ad^2$ is called the **domain map**.

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- In such a structure, the **domain algebra**

$$K_d := d(K) = \{\, x \in K | d(x) = x\}$$

is a Boolean algebra $(K_d,\ +,\ 0,\ \cdot,\ 1,\ \neg)$ where $\neg = ad_{|K_d}$.

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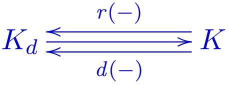
- We similarly axiomatise a notion of **codomain** $r : K \to K$.
- K is a **(Boolean) modal Kleene algebra (MKA)** when equipped with a domain and a codomain map.



Modalities and conversion

Let K be a modal Kleene algebra and $x \in K$.

- We define the **forward diamond operator** given by x:

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  Finally, conversion is **contracting** in the sense that

$$d(x) \leq x \cdot \overline{x}.$$

Rewriting properties in MKAs

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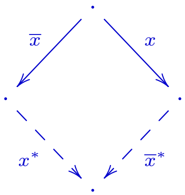

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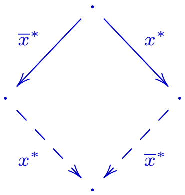

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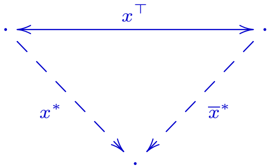

where $x^\top = (x + \overline{x})^*$ is the **equivalence** generated by x.

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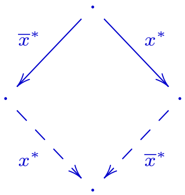

- We say that x is **convergent** when it both terminates and is confluent.

An algebraic formalisation of coherence

2-Kleene algebras

- A **globular 2-Kleene algebra** is a tuple

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- Multiplications satisfy the **weak exchange law**:

$$(A \odot_1 A') \odot_0 (B \odot_1 B') \leq (A \odot_0 B) \odot_1 (A' \odot_0 B').$$

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- (co-)Domains satisfy **absorption laws** $d_1 \circ d_0 = d_0$ and $r_1 \circ r_0 = r_0$. The domain algebra K_{d_i} will be denoted by K_i.

2-Kleene algebras

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$$(K,\ +,\ 0,\ \odot_i,\ 1_i,\ (-)^{*_i},\ d_i,\ r_i)_{i\in\{0,\ 1\}}$$

  - $(K,\ +,\ 0,\ \odot_0,\ 1_0,\ (-)^{*_0})$ is a Boolean MKA with converse.
  - $(K,\ +,\ 0,\ \odot_1,\ 1_1,\ (-)^{*_1})$ is an MKA.

- The 1-star is a **lax morphism** w.r.t. 0-multiplication by 1-dimensional elements, *i.e.* for all $A \in K$ and $\phi \in K_1$,

$$\phi \odot_0 A^{*_1} \leq (\phi \odot_0 A)^{*_1}, \quad \text{and} \quad A^{*_1} \odot_0 \phi \leq (A \odot_0 \phi)^{*_1}.$$

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- Finally, the (co-)domains satisfy **globularity conditions**:

$$d_0 \circ d_1 = d_0 \ \text{ and } \ d_0 \circ r_1 = d_0, \quad d_1(A \odot_0 B) = d_1(A) \odot_0 d_1(B),$$
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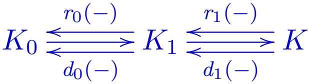

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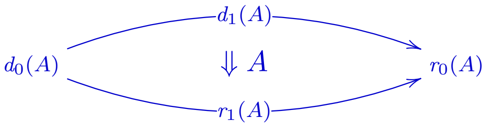
- An element $A \in K$ is represented graphically by:



Example: 2-path algebra

- Perhaps not enough time...

Paving via modalities

- Just as in the case of MKAs, we obtain **modalities** with respect to each multiplication:

$$\begin{aligned} |A\rangle_0 : K_0 &\longrightarrow K_0 \\ p &\longmapsto d_0(A \odot_0 p) \end{aligned} \qquad \begin{aligned} |A\rangle_1 : K_1 &\longrightarrow K_1 \\ \phi &\longmapsto d_1(A \odot_1 \phi) \end{aligned}$$

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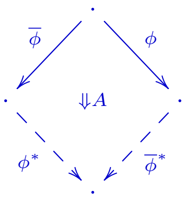

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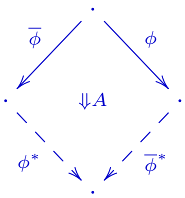

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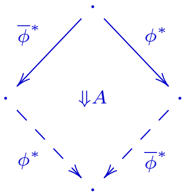

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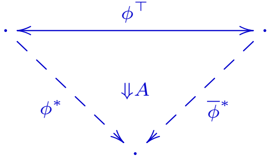

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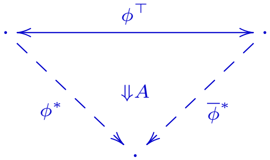

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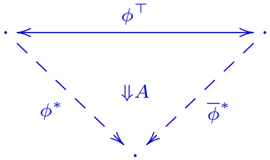

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- The **completion** of A is the element $\hat{A}^{*_1}$.

Rewriting strategies and coherence

Rewriting strategies

Let K be an MKA, $x \in K$ and $p \in K_d$.

- The **equivalence** generated by x is the element $x^\top := (x + \overline{x})^*$.
- The x-**saturation** of p is the element $|x^\top\rangle(p) \in K_d$.
- A **covering set** for x is an element $q \in K_d$ such that $|x^\top\rangle(q) \geq 1$.
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- A **skeleton** of x is a minimal wide sub.

- Given a section s_0 of x, a **strategy for x relative to s_0** is:
  - a skeleton σ of $x^\top s_0$,
  - such that $s_0 \sigma \leq s_0$.

Convergence yields a strategy

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Proposition

Let K be a modal Kleene algebra and $x \in K$. If x is convergent, then

- *nf_x is a section of x.*
- *A skeleton σ of $exh(x)$ is a strategy for x with respect to nf_x.*

Abstract coherence theorem

Theorem (Coherent normalising Newman's lemma)

Let K *be a globular* 2*-Kleene algebra such that*

- $(K_0, +, 0, \odot_0, \mathbb{1}_0, \neg)$ *is a complete Boolean algebra,*
- K_1 *is continuous with respect to* 0*-restriction.*

Let $\phi \in K_1$ *be convergent and* σ *be a skeleton of* $exh(\phi)$*. We have*

$$\overline{\phi} \odot_0 \phi \leq |A\rangle_1(\phi^{*0} \odot_0 \overline{\phi}^{*0}) \qquad \Rightarrow \qquad \overline{\phi}^{*0} \odot_0 \phi^{*0} \leq |\hat{A}^{*1}\rangle_1(\sigma \odot_0 \overline{\sigma})$$

A *is a local confluence filler for* ϕ

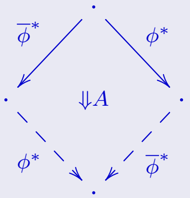
$\hat{A}^{*1}$ *paves branchings in* ϕ *to confluences in* σ

Abstract coherence theorem

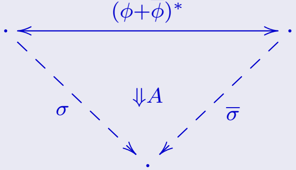
Theorem (Abstract coherence theorem)

Let K be a Boolean globular 2-Kleene algebra satisfying the previous hypotheses and $\phi \in K_1$ convergent and a skeleton σ of $exh(\phi)$. We have

$$\overline{\phi}^{*_0} \odot_0 \phi^{*_0} \leq |A\rangle_1(\phi^{*_0} \odot_0 \overline{\phi}^{*_0}) \qquad \Rightarrow \qquad \phi^{\top_0} = (\phi + \overline{\phi})^{*_0} \leq |\hat{A}^{*_1}\rangle_1(\sigma \odot_0 \overline{\sigma}).$$



$\Rightarrow$



A is a local confluence filler for ϕ

$\hat{A}^{*_1}$ paves zig-zags in ϕ to confluences in σ.

Conclusion

- We have expressed and proved a coherence theorem for ARS in the algebraic context of globular 2-Kleene algebras.
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- The end goal for ARS is a formalisation of **cofibrant replacement** via convergent rewriting.
- After finishing the case of ARS, move on to **string and term** rewriting systems:
  - Capture **contexts** via residuation,
  - thereby state and prove the **critical branching lemma**.

Thank you