Update on semisimplicial types in homotopy type theory

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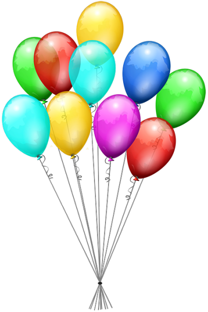
A ca. 10 year old open problem

A brief history:

- ca. 2005–06: Early beginnings of HoTT/UF
- 2012–13: The IAS special year on Univalent Foundations of Mathematics.
- 2012-11-12: Lumsdaine's notes on the problem at the IAS HoTT wiki.
- 2013-02-25: Voevodsky introduces HTS at Joyal's 70$^{\text{th}}$ birthday conference
- ca. 2013: Herbelin gave a detailed study, published in MSCS 2015.
- 2014-03-03: Shulman says "Homotopy type theory should eat itself"
- 2015-06-30: Poll at 1$^{\text{st}}$ HoTT/UF Workshop in Warsaw: 80% of attendees believe problem is unsolvable.
- Some proposed solutions: 2018 Finster and 2021 Campion.



Recent progress on metatheory of HoTT

Recently solved problems:

1. Homotopy canonicity, Kapulkin–Sattler 2019.
2. Models in all $(\infty, 1)$-toposes, Shulman 2019.
3. A type theoretic model structure on cubical sets equivalent to spaces, Awodey–Cavallo–Coquand–Riehl–Sattler 2019.
4. Normalization for cubical type theory, Sterling–Angiuli 2021.

Some still open problems:

- Stronger internal language theorems.
- Constructive version of 2, presumably with the help of 3.
- The *coherence problem* (CP), subject of this talk.

(Spoiler: Still open!)

Motivation for the problem

HoTT/UF aims to be a foundation for all mathematics (constructive or classical, by assuming the axiom of choice, sets cover, and Whitehead's principle).

Structure Identity Principle

Each notion of mathematical object comes with a natural notion of equivalence. When expressed in type theory, the notion should be a type A whose identity types express this notion of equivalence.

As a consequence, if $a, b : A$ can be identified (we have $p : a =_A b$), then a, b share the same structural properties.

Earlier: Makkai's work towards Invariant/Categorical Foundations (which was to be based on higher categories).
Examples: Strict 1-categories (form a 1-type) vs 1-categories (form a 2-type).
NB Voevodsky thought the main insight of UF was to base the theory on ∞-*groupoids*, not higher categories. The failure to solve CP in a sense undermines this.

What is the coherence problem, I

So what is CP: *Roughly* speaking, it's the task of developing the theory of $(\infty, 1)$-categories in HoTT, enabling the usual theorems to be proved, following the SIP, and such that each universe $\mathcal{U}$ can be equipped with the structure of an $(\infty, 1)$-category with hom-types equivalent to function types.

Natural approach: Define the types of semisimplicial types. This would do the trick, following Capriotti–Kraus 2017, Kraus 2022. And it's very tantalizing, because we can write down the *externally* finite cases $0, 1, 2, \ldots$ (see below).

Further correctness criteria:

- With CP solved, we in any case get the truncated types of semisimplicial types, $\mathrm{PSh}(\mathbb{\Delta}_{\leq n})$. We should require the above for fixed external n.
- Using the $(\infty, 1)$-topos semantics, it should be the case that the interpretation of the type of categories is equivalent to the type of $(\infty, 1)$-category objects in $\mathcal{E}$, for each $(\infty, 1)$-topos $\mathcal{E}$.

Comparison with other open problems

Two aspects:

- A concrete mathematical problem is given by a type A. (The problem is: give an element $a : A$) – It's desirable that A be a proposition.
  Q: Can we give a type/proposition P that captures CP? – This seems just as hard as CP itself! (But is it?)
- Usually, when we have an open problem P (e.g., RH, P vs. NP, ...) that people have unsuccessfully tried to solve, they tend to conclude that it's hard, but not necessarily that it's independent of the foundational setup.
  In contrast, this seems to be a common opinion regarding the CP (again, see below).

Evidence from homotopy theory and higher algebra

Three observations:

- We can express most of "classical" (i.e., before ca. 1970) homotopy theory synthetically, but the modern theory is elusive as it relies strongly on higher categorical methods.
  Recent example: EHP sequence and
  $\pi_1(\mathbb{S}^2 \to \mathbb{S}^2, f) \simeq \mathbb{Z}/2|\deg(f)|\mathbb{Z}$ (Cagne–B–Kraus–Bezem)
- We know how to define ∞-groups: They are pointed, connected types. Note that ∞-groups are precisely group-like ∞-monoids (A_∞-types). We don't know how to define A_∞-types!
  This is of course related to the fact that our foundation is based on higher groupoids, not higher categories.
- We know how to define the type of $\mathbb{S}$-module spectra: They are just spectra. We don't know how to define the type of $\mathbb{Z}$-module spectra.

Cell complexes

A related problem: Define the geometric realization map $\mathrm{sSet} \to \mathcal{U}$.

We have a good definition of cell complexes: Define $\mathrm{CC} : \mathbb{N} \to \mathcal{U}$ simultaneously with $[\![\text{-}]\!] : \prod_{n:\mathbb{N}}(\mathrm{CC}(n) \to \mathcal{U})$ by induction:

- $\mathrm{CC}(0) :\equiv \mathrm{Set}$, $[\![A]\!] :\equiv A$.
- $\mathrm{CC}(n+1) :\equiv \sum_{X:\mathrm{CC}(n)} \sum_{A:\mathrm{Set}} (A \times \mathbb{S}^n \to [\![X]\!])$, $[\![(X, A, \alpha)]\!]$ is the pushout:

$$\begin{tikzcd} A \times \mathbb{S}^n \arrow[r, "\alpha"] \arrow[d] & {[\![X]\!]} \arrow[d] \\ A \arrow[r] \arrow[ur, phantom, "\ulcorner"] & {[\![(X, A, \alpha)]\!]} \end{tikzcd}$$

NB We have a version of $\mathbb{HP}^\infty$ as a simplicial set. If we have a realization map $\mathrm{sSet} \to \mathrm{CC}$, we get the type $\mathbb{HP}^\infty$.

Diagram types we can construct

Fix a universe $\mathcal{U}$.

We do have some classes of $(\infty, 1)$-categories C for which the presheaf type $\mathrm{PSh}(C)$ is definable:

- C is an ∞-groupoid, i.e., a type: $\mathrm{PSh}(C) = (C \to \mathcal{U})$.
- C is a direct $(\infty, 1)$-category of *externally* finite height (see next slide).
- C is free on a graph.
- C is the globe category.

Presheaves on direct categories

(Cf. Shulman 2017 and Kraus and Sattler 2017.)

Take as *Ansatz* that we've solved CP: We can define *internally* finite height direct categories by induction, $\mathrm{Dir} : \mathbb{N} \to \mathcal{U}$.

- $\mathrm{Dir}(0) :\equiv \mathcal{U}$, $\mathrm{PSh}(A) :\equiv (A \to \mathcal{U})$ for $A : \mathcal{U}$.
- $\mathrm{Dir}(n+1) :\equiv \sum_{C:\mathrm{Dir}(n)} \sum_{A:\mathcal{U}} (A \to \mathrm{PSh}(C))$ (notation: $C +_\partial A$), with $\mathrm{PSh}(C +_\partial A) :\equiv \sum_{X:\mathrm{PSh}(C)} \prod_{a:A} (\mathrm{Hom}_{\mathrm{PSh}(C)}(\partial a, X) \to \mathcal{U})$.

Intuition: Specifying the morphisms from an object of C to an object $a : A$ amounts to specifying the representable presheaf $\mathrm{Hom}(\text{-}, a)$ restricted to C, and we call this ∂a. A presheaf on $C +_\partial A$ consists of a presheaf X on C and, for every object $a : A$ of top height, the elements at a indexed by the universal restriction to the lower level:

$$\begin{tikzcd}
\partial a \arrow[r, hook] \arrow[d] & a \arrow[d, dashed] \\
X \arrow[r, hook] & (X, X')
\end{tikzcd}$$

Example: Presheaves on $[n]$

In special cases, we can independently define the Hom-type, e.g., $[n] = (0 < \cdots < n)$, where $\partial n = 1$ (terminal object), so $\mathrm{Hom}(1, X) = \Gamma X$ (global sections).

Three solutions:

- (contexts) Define $\mathrm{PSh}([\text{-}]) : \mathbb{N} \to \mathcal{U}$ simultaneously with Γ:
  $\mathrm{PSh}([n+1]) \coloneqq \sum_{X:\mathrm{PSh}([n])} (\Gamma X \to \mathcal{U})$ and
  $\Gamma(X, X') \coloneqq \sum_{x:X} X'(x)$.
- (telescopes) $\mathrm{PSh}([n+1]) \coloneqq \sum_{A:\mathcal{U}} (A \to \mathrm{PSh}([n]))$.
- ($[n]$ is free on a graph)
  $\mathrm{PSh}([n]) \coloneqq \sum_{X:\mathrm{Fin}(n+1)\to\mathcal{U}} \prod_{k:\mathrm{Fin}(n+1)\setminus\{\top\}} (X(k+1) \to X(k))$.

Example: $\mathrm{PSh}(\mathbb{G})$

$\mathbb{G}$ is the globe 1-category, $\mathbb{G}_{\leq n} = (0 \rightrightarrows 1 \rightrightarrows \cdots \rightrightarrows n)$, where $\sigma\sigma = \tau\sigma$ and $\sigma\tau = \tau\tau$. Here, $\partial(k+1)$ is the k-globe boundary.

Two solutions (that I know of!):

- Define $P(n) = \mathrm{PSh}(\mathbb{G}_{\leq n})$ simultaneously with $B_n : P(n) \to U$:
  $P(n+1) :\equiv \sum_{X:P(n)} (B_n(X) \to \mathcal{U})$ and
  $B_{n+1}(X, X') :\equiv \sum_{b:B_n(X)} X'(b) \times X'(b)$
- Define $P(n+1) :\equiv \sum_{A:\mathcal{U}} (A \times A \to P(n))$.

Unfortunately, this doesn't solve CP, since we can't express $(\infty, 1)$-categories as globular types satisfying a *property*, only equipped with *extra structure*.

The problematic case: $\mathrm{PSh}(\mathbb{\Delta})$

Here, by $\mathbb{\Delta}$ is mean the semisimplex category: The objects are natural numbers and maps from n to m are strictly increasing maps $[n] \to [m]$. Let $\mathbb{\Delta}_{\leq n} = ([0] \rightrightarrows [1] \cdots [n])$.

The problem is that there doesn't seem to be a simple, uniform definition of $\mathrm{Hom}(\partial\Delta^n, X)$!

$$
\begin{aligned}
\mathrm{PSh}(\mathbb{\Delta}_{\leq 0}) &= \mathcal{U} \\
\mathrm{PSh}(\mathbb{\Delta}_{\leq 1}) &= \sum_{X_0 : \mathrm{PSh}(\mathbb{\Delta}_{\leq 0})} (X_0 \times X_0 \to \mathcal{U}) \\
\mathrm{PSh}(\mathbb{\Delta}_{\leq 2}) &= \sum_{(X_0, X_1) : \mathrm{PSh}(\mathbb{\Delta}_{\leq 1})} \\
&\qquad \left(\Big(\sum_{(x_0, x_1, x_2) : X_0 \times X_0 \times X_0} X_1(x_0, x_1) \times X_1(x_0, x_2) \times X_1(x_1, x_2) \Big) \to \mathcal{U} \right) \\
&\quad \vdots
\end{aligned}
$$

Idea: Generalize

To analyze this problem, consider again the definition above of presheaves on general direct categories.

$$\begin{aligned}&\mathrm{Hom}_{\mathrm{PSh}(C+_\partial A)}((X,X'),(Y,Y'))\\ &\qquad = \sum_{f:\mathrm{Hom}_{\mathrm{PSh}(C)}(X,Y)} \prod_{a:A} \prod_{x:\mathrm{Hom}_{\mathrm{PSh}(C)}(\partial a, X)} (X'(x) \to Y'(f \circ x))\end{aligned}$$

Of course, the composition $f \circ x$ refers to the 2-cells in $\mathrm{PSh}(C)$.

In general, n-cells in $\mathrm{PSh}(C +_\partial A)$ will depend on $(n+1)$-cells in $\mathrm{PSh}(C)$ as well as n-cells in $\mathcal{U}$.

Idea: Maybe we can define all of this simultaneously, representing n-cells in $\mathcal{U}$ via semisimplicial structure.

Problem: The dependency/indexing doesn't match!

General idea: Coherence bootstrap

We could hope to define some notion of n-approximate coherence, and make use of the n-approximate coherence of the universe to define the notion of $(n+1)$-approximate coherence.

What is the right notion of n-approximate coherence? A semisimplicial diagram? An n-approximate model of type theory of some sort? (Cf. Shulman's *autophagia*) – Perhaps using Kraus' ∞CwFs?

Possible problem: Universe inconsistency.

Functional relations in the universe

Another problem: How are we going to represent the n-cells in $\mathcal{U} = \mathrm{PSh}(1)$?

It seems promising to use *functional relations* as presheaves over the posets $\mathbb{\Delta}/\Delta^n$, equivalently, of non-empty finite subsets of $\mathrm{Fin}\, n$ (or general finite sets).

Because of the equivalence $\mathrm{PSh}(\mathbb{\Delta}/\Delta^n) \simeq \mathrm{PSh}(\mathbb{\Delta})/\Delta^n$, we can express the functional relations as covariant families.

Variation: Cubical diagrams

We are led to consider the related problem of defining cubical diagrams, i.e., the types of commuting n-cubes for all n. (Recover simplicial relations by requiring corner be a contractible type.)

We again need to define the projection/evaluation maps for such cubical diagrams, and we need that these are coherent, so maybe we can encode the coherence of a cubical diagram as a family of cubical diagrams?
(I couldn't get this to work.)

Other idea variations

Some variations on the above ideas:

- It seems useful to be able to evaluate a cubical/simplicial face diagram at a combination of faces. We can consider the full face lattice of Δ^n, then recover $\mathrm{PSh}(\mathbb{\Delta}/\Delta^n)$ as the *sheaves* on this.
- The objects of the face lattice are all joins of principal faces: Perhaps we should consider $\mathbb{\Delta}_{\mathrm{a}}/\Delta^n$ as a multicategory wrt joins, and then define the type of *multipresheaves*?
- This leads to: Is there a notion of directed polynomial monad fitting the question mark:

$$
\begin{tikzcd}
\mathrm{Cat} \arrow[r, hook] & \mathrm{PolyMnd} \\
\mathrm{DirCat} \arrow[r, hook] \arrow[u, hook] & ? \arrow[u, hook]
\end{tikzcd}
$$

Negative solution?

From *Type theory should eat itself*:

> *(Sattler:) There ought to be some kind of metatheorem making precise the intuition that they don't exist.*

(Shulman:) Yes, I occasionally wonder about this as well. Sometimes it feels kind of like how natural numbers objects (or more generally infinite constructions) don't necessarily exist in an elementary topos, because the category of finite sets is such. But of course in this case we are assuming lots of infinite constructions, including the natural numbers, and there are even some infinite structures of type dependency that we can define easily, like towers and globular types. Somehow it feels like what's different about semisimplicial types is that the "complexity" also diverges as we go to infinity, rather than staying constant, but it's unclear to me how to make that precise, or construct a model in which structures of "divergent complexity" don't exist even though some infinite structures do.

Conclusions

Takeaways:

- CP is *the* major problem keeping HoTT/UF from being a self-sustaining fully-fledged foundation.
- 10 years is not *that* long, in the grand scheme of things, for a problem to remain open, but this one is quite annoying: Please help!
- Poll: Do *you* think its unsolvable?

Thanks for your attention!