Weak ω-categories as models of a type theory

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CEA LIST

Logic and higher structures
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Dependent type theories and higher structures

HoTT and weak ω-groupoids

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- This suggest a link between dependent type theories and higher structures.

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In this type theory, the identity types are not inductive, instead there is a family of term constructors that witnesses the algebraic structure.

DTT and higher structures

The correspondence between dependent type theories and higher algebraic structure follows the principle

$$\text{type dependency} \rightsquigarrow \text{higher dimensional shapes}$$
$$\text{term constructors} \rightsquigarrow \text{algebraic structure}$$

Weak ω-categories

An Overview

Weak ω-categories are higher structure with directed arrows in all levels

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Lots of other definitions, with other shapes.

Weak ω-categories

The Grothendieck-Maltsiniotis definition

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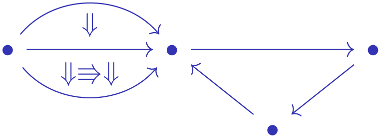
  Intuition : enforce a privileged direction on the rules
- Proven equivalent to Batanin-Leinster definition by Ara [1]

A globular definition

- Supported by globular sets



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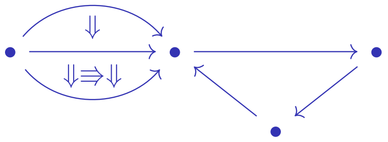

- Presheaf category whose representables are disks

Pasting schemes

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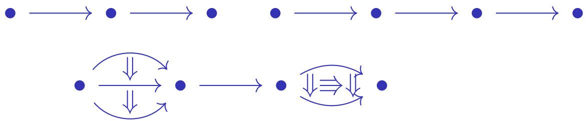

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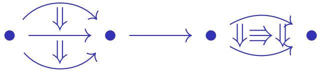

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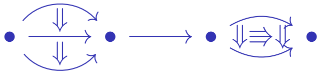

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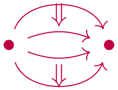

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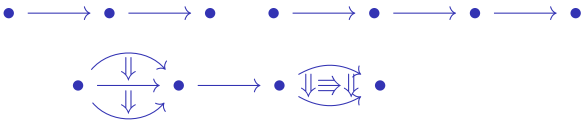

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\begin{array}{ccccccc}
D^{i_1} & & D^{i_2} & & \dots & & D^{i_k} \\
& \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & & \nwarrow \quad \nearrow & \\
& D^{j_1} & & D^{j_2} & \dots & D^{j_{k-1}} &
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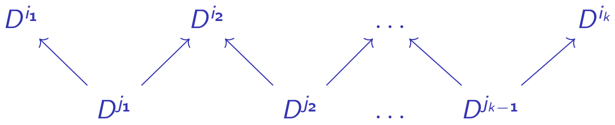

- The globular sums are exactly the pasting schemes.
  Define Θ_0 to the full subcategory of globular sets whose objects are the globular sums.

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actually one has to do inifitely many steps to build Θ_∞

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- We need to require those presheaves to preserve globular sums, to avoid having too much shapes allowed.

The type theory CaTT

Intuition

- Introduced by Finster and Mimram [4]
  Intuition : It defines the following "pushout"

$$\begin{array}{ccc}
\text{Grothendieck's } \omega\text{-groupoids} & \xrightarrow{\text{direction}} & \text{G.-M. } \omega\text{-categories} \\
\text{type theory} \downarrow & & \downarrow \text{type theory} \\
\text{Brunerie's type theory} & \xrightarrow[\text{direction}]{} & \text{CaTT}
\end{array}$$

The type theory CaTT

Dependent type theories and their categorical semantics

Building blocks of a dependent type theory (DTT)

A dependent type theory $\mathcal{T}$ has syntactic objects :

- Contexts $\Gamma, \Delta, \ldots$: lists of pairs of variables and types

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- Substitutions $\gamma, \delta, \ldots$: lists of pairs of variables and terms

$$\Delta \vdash \gamma : \Gamma$$

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The dependent type theories the structure in common

- A variable is a valid term in a context :

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$$\frac{\Delta \vdash \gamma : \Gamma \qquad \Gamma \vdash A \qquad \Delta \vdash t : A[\gamma] \qquad x \notin \Gamma}{\Delta \vdash \langle \gamma, x \mapsto t \rangle : (\Gamma, x : A)}$$

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A CwF is the collection of all this data

This presentation follows the style of Awodey's natural models

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- There is a CwF structure on the category of sets
- A model of the theory $\mathcal{T}$ is a morphism of CwF $\mathcal{S}_{\mathcal{T}} \to \mathrm{Set}$

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Presentation of the theory

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- Denote GSeTT the theory with just these type constructors

- $\mathcal{S}_{\mathsf{GSeTT}}$ is the opposite of finite globular sets
  For instance, the following context and globular sets are in correspondence

  `(x:*, y:*, z:*, f:x->y, g: y->z)` $\qquad \overset{x}{\bullet} \longrightarrow \overset{y}{\bullet} \longleftarrow \overset{z}{\bullet}$

Ps-contexts

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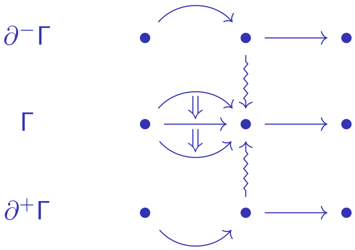
- Each ps-context Γ has a source $\partial^{-}\Gamma$ and a target $\partial^{+}\Gamma$



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$$\begin{aligned} \mathrm{Var}(t : A) &= \mathrm{Var}(\partial^-(\Gamma)) \\ \mathrm{Var}(u : A) &= \mathrm{Var}(\partial^+(\Gamma)) \end{aligned}$$

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- **Every two compositions of the same pasting scheme are related**

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \Gamma \vdash t : A \qquad \Gamma \vdash u : A}{\Gamma \vdash \mathsf{coh}_{\Gamma, t \underset{A}{\longrightarrow} u} : t \underset{A}{\longrightarrow} u} \qquad \begin{array}{ll} \mathrm{Var}(t : A) & = \mathrm{Var}(\Gamma) \\ \mathrm{Var}(u : A) & = \mathrm{Var}(\Gamma) \end{array}$$

Applying operations and coherences

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- We get terms in generic context by action of substitutions
  Hence relax the previous rules to have

$$\frac{\Gamma \vdash_{\mathsf{ps}} \qquad \partial^{-}\Gamma \vdash t : A \qquad \partial^{+}\Gamma \vdash u : A \qquad \Delta \vdash \gamma : \Gamma}{\Delta \vdash \mathsf{op}_{\Gamma, t \underset{A}{\to} u}[\gamma] : t[\gamma] \to u[\gamma]}$$

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(keeping the side condition)

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  Consider the ps-context
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  Both sides use all the variables of Γ_a (some are implicit).

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  Consider the ps-context
  $\Gamma_c = (x : \star, y : \star, f : x \to y, z : \star, g : y \to z)$.
  We have $\partial^- \Gamma_c = (x : \star)$ and $\partial^+ \Gamma_c = (z : \star)$.
  So we deduce the term $\Gamma_c \vdash \mathrm{op}_{\Gamma_c, x \to z} : x \to z$ (denoted $\mathtt{comp}$).

- Associativity :
  Consider the ps-context
  $\Gamma_a = (x : \star, y : \star, f : x \to y, z : \star, g : y \to z, w : \star, h : z \to w)$.
  We have the type
  $\Gamma_a \vdash \mathtt{comp}(f, \mathtt{comp}(g, h)) \to \mathtt{comp}(\mathtt{comp}(f, g), h)$.
  Both sides use all the variables of Γ_a (some are implicit).
  So we deduce the term $\Gamma_c \vdash \mathrm{op}_{\Gamma_c, x \to z} : x \to z$ (denoted $\mathtt{comp}$).

The type theory CaTT

Semantics of the theory

The subcategory of ps-contexts

Define PS : the full subcategory of $\mathcal{S}_{\mathsf{CaTT}}$ whose objects are the ps-contexts

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Theorem (B., Finster, Mimram)

The category PS is equivalent to $\Theta_\infty^{\mathrm{op}}$

Models of the theory

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The models of CaTT are equivalent to the G.-M. weak ω-categories

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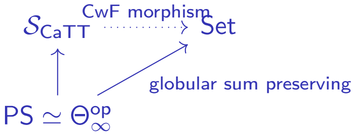

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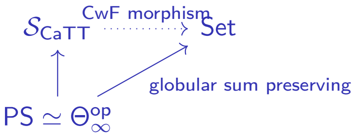

Proved by showing the initiality theorem for the theory CaTT.

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- There is work conducted around this conjecture and extension of CaTT.
  Ongoing work related to this question and CaTT by Finster, Vicary, Markakis, Rice

Thank you !

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