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Subtitles for

SCHROEDINGER'S CUT

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1 – THE LOGICAL JAILBREAK

- No *need* of systems : Gentzen, Herbrand.

  BHK : proofs as *functions,* leaks (realisability).

  Absurdity : empty set (initial object).

  Tertium non datur : $\neg A \vee \neg\neg A$.

- Promethean hypothesis : logic *(= pure reason)* makes sense !

  Non emptiness : everything has a « proof » including absurdity.

  No conviction carried by « proofs » albeit essential to architecture.

  Invisible files: `.mailrc`, etc. make computer work.

- Invisible proofs appear in *usine* (synthetic *a posteriori*).

	Analytic	Synthetic
Explicit	Constat	Usine
Implicit	Performance	Usage

2 – USINE : MISSING PROOFS

- **Involutivity of negation :** ***topsy-turvying*** **of proofs.**

Several conclusions **hard to handle.**

Proof-net **criterion, switchings (Herbrand).**

- **Atoms of proof $\mathcal{P}$ of A linked.**

Upstairs : **identity links $[\![\, p, \sim p \,]\!]$.**

Downstairs : **switching links $[\![\, p_1, \ldots, p_k \,]\!]$.**

Wild **proof $\mathcal{S}$ of $\sim A$.**

Correctness : ***cut*** **between $\mathcal{P}, \mathcal{S}$ normalises.**

- **Alternative proofs** ***(counterproofs)*** **form the real logical world.**

Hypothesis : **may be quantum-like.**

Dubious **but leads to C^**-algebraic* approach ; stimulating interaction.**

3 – C^*-ALGEBRAS

- C^*-algebras generalised *matrix algebras.*

  Involutive complex Banach (complete w.r.t. norm) algebra.

  $(u+v)^* = u^*+v^*$, $(\lambda u)^* = \bar{\lambda} u^*$, $(uv)^* = v^* u^*$, $u^{**} = u$, $1^* = 1$

  C^*-norm : $\|uu^*\| = \|u\|^2$.

- Spectrum $\mathrm{Sp}(u) := \{\lambda \in \mathbb{C}; u - \lambda \text{ not invertible}\}$

  Eigenvalues : $\mathrm{Sp}(u)$ compact and *non empty.*

  If u ***normal*** i.e., $uu^* = u^* u$, then $\|u\| = \sup\{|\lambda|; \lambda \in \mathrm{Sp}(u)\}$.

  Algebra generated by *normal* u isomorphic to $\mathcal{C}(\mathrm{Sp}(u))$.

  u corresponds to inclusion map $\mathrm{Sp}(u) \hookrightarrow \mathbb{C}$.

  Applies to : *hermitians* $(u = u^*)$ and *unitaries* $(uu^* = u^* u = 1)$.

- Since uu^* is hermitian, norm *algebraically* definable.

  $$\|u\| = \|uu^*\|^{1/2}$$

  Quotients : sole way to diminish *spectrum,* hence the norm.

  Greatest stellar norm must be chosen in case of doubt.

4 – THE ALGEBRA $\mathbb{U}$

- **Fixed functional language ; *irrelevant* provided contains $g(\cdot,\cdot)$ and a**

  Hilbert space : ℓ^2 generated by closed terms.

  Term t operates by *substitution* $\ell^2 \otimes \ldots \otimes \ell^2 \mapsto \ell^2$.

  Example : $[g(x,y)](\sum \lambda_i \cdot t_i \otimes u_i) = \sum \lambda_i \cdot g(t_i,u_i)$, hence

  $$[g(x,y)]^*(3 \cdot g(t,u) - 2 \cdot g(u,t) + a) = 3 \cdot t \otimes u - 2 \cdot u \otimes t.$$

- **Involutive algebra $\mathbb{U}$ spanned by *monomials* tu^*.**

  Variables : bound (local to monomial) ; the same in t, u.

  Composition : $tu^* \cdot vw^* := t\theta w\theta^*$ if u, v *unifiable* with *mgu* θ.

  Otherwise $tu^* \cdot vw^* := 0$.

  Example : $xx^* \cdot tu^* = tu^*$.

  Involution : $(tu^*)^* := ut^*$; *contravariant* compositionwise.

- **Algebra $\mathbb{U}$ *operates* on ℓ^2, e.g., $tu^* : \ell^2 \mapsto \ell^2$.**

  Several C^*-norms, e.g., $g(x,y)g(x,y)^* + aa^* = 1$.

- **Keep $\mathbb{U}$ *algebraic* including complex coefficients.**

5 – Geometry of interaction

- $[\![a, b, c]\!]$ **becomes** ***permutation*** $\tau(a) := b, \tau(b) := c, \tau(c) := a.$

  Correctness **opposes proof** σ **and** ***counterproof*** τ.

  Cyclicity : $(\sigma\tau)^n \neq id$ **for** $0 < n < N.$

  Application $[\varphi]\alpha$ **better handled in terms of** ***matrices.***

$$[\varphi]\alpha := (1-\alpha\alpha^*)(\varphi+\varphi\alpha\varphi+\varphi\alpha\varphi\alpha\varphi+\ldots+\varphi(\alpha\varphi)^N)(1-\alpha^*\alpha)$$

  Provided $\alpha\varphi$ ***nilpotent :*** $(\alpha\varphi)^{N+1} = 0.$

- ***Partial isometries :*** $uu^*u = u$ **(source** $u^*u,$ **target** uu^***).**

$$[\varphi]\alpha := (1-\alpha\alpha^*)\varphi(1-\alpha\varphi)^{-1}(1-\alpha^*\alpha)$$

  Equation : $\varphi(y + x) = \alpha(y) + x'$ **(input** x, **output** x', **feedback** y**).**

- **Symmetric** $A/\sim A$ **but fails to explain** ***invisibility.***

- ***Partitions*** **(Danos-Regnier) better on that issue.**

  Links **arbitrary arity** ≥ 1.

  Bipartite **graph made from** σ, τ **with atoms as edges.**

  Connected and acyclic : σ, τ **not both binary** ***(Euler-Poincaré).***

6 – Proofs as projections

- **Partition** $\rightsquigarrow$ ***subspace,*** **e.g.,** $\{\lambda a + \mu(b - c)\}$ **for** $\{\{a\}, \{b, c\}\}$.

  Projection $(u = u^2 = u^*)$ **correspond to *target* subspace** $|u|$.

  $[\![t_1, \ldots, t_n]\!]$ **becomes** $(\lambda_1 t_1 + \ldots + \lambda_n t_n)(\lambda_1 t_1 + \ldots + \lambda_n t_n)^*$.

  Novelty : (algebraic) complex coefficients.

- **« Graphlike » composition (cut-elimination).**

  Modus Ponens between subspaces $|[\varphi]\alpha| := (|\varphi| + |\alpha|) \cap |\alpha|^\perp$

  Feedback : $|[\varphi]\alpha| := \{y \in |\alpha|^\perp \; ; \; \exists x \in |\alpha| \;\; -x + y \in |\varphi|\}$.

  Coherent spaces : $[\varphi]\alpha := \{b \; ; \; \exists a \in \alpha \; (a, b) \in \varphi\}$.

- ***Intersection*** $u \cap v$ **as « limit » of decreasing sequence**

  $u \geq uvu \geq (uvu)^2 \geq \ldots \geq (uvu)^n \geq \ldots$

  Provided $\mathrm{Sp}(uv) \subset \{\lambda_1, \ldots, \lambda_n, 1\} \;\; (0 \leq \lambda_i < 1)$ **equals**

  $uvu \cdot (1 - \lambda_1)^{-1}(uvu - \lambda_1) \cdot \ldots \cdot (1 - \lambda_n)^{-1}(uvu - \lambda_n)$

  Dually : $u \uplus v := 1 - ((1 - u) \cap (1 - v))$ **closed linear span of *union*.**

  $[\varphi]\alpha := (\varphi \uplus \alpha) \cap (1 - \alpha)$, **provided** $\varphi \cap \alpha = 0$.

7 – A LOGICAL CHALLENGE

- **Physical world makes logical « mistakes. »**

  Belief : world of thought without relation to physics.

  Object/subject : truth values, axiomatics.

  Quantum « logics » sort of *flogging* à la Xerxes (Herodotus VII, 34–35).

- **(Invisible) proofs as *wave functions*.**

  Measurement : cut-elimination.

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- **Challenge : *analytic* (performance) rather than synthetic (purely logical).**

  Constat : find the right objects.

  Performance : and the right (non deterministic) evaluation procedure.

8 – TRANSCENDENTAL SYNTAX, REVISITED

- ***Transcendental syntax*** **succeeds w.r.t. usual logic.**

  Stars $[\![\, t_1, \ldots, t_n \,]\!]$ **and *constellations*.**

  Normalisation based on (with θ m.g.u. of t_{m+1}, u_0).

$$[\![\, t_1, \ldots, t_{m+1} \,]\!] + [\![\, u_0, \ldots, u_n \,]\!] \rightsquigarrow [\![\, t_1\theta, \ldots, t_m\theta, u_1\theta, \ldots, u_n\theta \,]\!]$$

  Restriction : *rays* $[\![\, t_1, \ldots, t_n \,]\!]$ have exactly the *same* (bound) variables.

- ***Asystemic*** **(non axiomatic) reconstruction of logic : equality, arithmetic.**

  Truth of *proofs* based on *Euler-Poincaré.* « False » ones invisible.

  Kills any kind of truth table, including tarskian *pleonasm.*

- **Quantum input : see the linear *par* $A \parr B$ as *superposition* of A and B.**

  $A \parr B$ **do interact because *same* variables in $[\![\, t, u \,]\!]$.**

  Microscopic entities seen as *closed* stars : no way to *duplicate* them.

- ***Measurement :*** **illegal star $[\![\, t, u[x] \,]\!]$ with variables in $u[x]$, not in t.**

  Schrödinger's cat : result of cut between measurements.

  Yields absurd superposition $[\![\, u[x], v[y] \,]\!]$.

9 – Work in progress (?)

- Algebra $\mathbb{U}$ as the *analytic* space ; definite progress.

  Projections $(u = u^2 = u^*)$ replace constellations.

  $[\![t_1, \ldots, t_n]\!]$ becomes $(\lambda_1 t_1 + \ldots + \lambda_n t_n)(\lambda_1 t_1 + \ldots + \lambda_n t_n)^*$.

  Novelty : (algebraic) complex coefficients.

- *Quid* of « illegal » stars ?

  $[\![a, f(x)]\!]$ should be $1/2 \cdot (a + f(x))(a + f(x))^*$.

  Applied to aa^* would yield $f(x)f(x)^*$.

  Applied to bb^* would yield 0.

- With $f(x)f(x)^* - f(c)f(c)^* + 1/2 \cdot (a + f(c))(a + f(c))^*$

  Applied to aa^* yields $f(x)f(x)^*$.

  Applied to bb^* yields *lacunary* $f(x)f(x)^* - f(c)f(c)^*$.

- Questions : *lacunary* implies *invisible ?* (Object/Subject).

  Probabilities $\lambda\overline{\lambda}, \mu\overline{\mu}$ in case of $(\lambda \cdot a + \mu \cdot b)(\lambda \cdot a + \mu \cdot b)^*$, etc.

- Drop in quality : find analogue of $|[\varphi]\alpha| := (|\varphi| + |\alpha|) \cap |\alpha|^{\perp}$.

10 – A MISFIRE : QUANTUM COHERENT SPACES

- ***Coherent spaces*** **based on duality on** $\wp(|X|)$

  Duality based on *cardinal* : $\sharp(a \cap b) \leq 1.$

  Application $\sharp([F]a \cap b) = \sharp(F \cap (a \times b))$

  Identity $id := \{(x, x); x \in |X|\}$

  $$\sharp(id \cap a \times b) = \sharp(a \cap b)$$

- **Replace subsets by *hermitians* on finite dimensional** $\mathbb{C}^{|X|}$.

  Duality based on *trace* : $(0 \leq \mathrm{Tr}(h \cdot k) \leq 1).$

  Application $\mathrm{Tr}([F]h \cdot k) = \mathrm{Tr}(F \cdot (h \times k))$

  Identity $\sigma(x \otimes y) := y \otimes x$

  $$\mathrm{Tr}(\sigma \cdot (h \otimes k)) = \mathrm{Tr}(h \cdot k)$$

- **Remains a toy example : *finite* dimension is absolutey required.**

- ***Honest*** **semantics : works *against* prejudices.**

  « Bavure » $\forall \Rightarrow \exists \rightsquigarrow$ **escamotage du *témoin* gênant (modèle vide).**

- **La sémantique, c'est un peu l'*IGPN* de la logique.**