Proofs as meanings and semantics of proofs

Introduction

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Logic and Interactions
Week III - Logic and transdisciplinarity
CIRM, Marseille 7–11 February 2022

Proof-theoretic semantics

The central notion of a theory of meaning
is not that of truth
but rather that of proof

The BHK explanation

What is a proof of a logically complex sentence?

- a proof of $A \wedge B$ is a pair consisting of a proof of A and a proof of B
- a proof of $A \supset B$ is a function that maps any proof of A onto a proof of B
- a proof of $A \vee B$ is either a proof of A or a proof of B
  (together with a bit of information, telling which of the two is the case)
- Nothing is a proof of $\bot$
- $\top$ has a (unique) trivial proof

Meaning explanations

- A mathematical sentence expresses
  the “expectation” of a construction satisfying certain conditions

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Meaning explanations

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- A proof of a sentence

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Meaning explanations

- A mathematical sentence expresses a specification, i.e. a data type
- A proof of a sentence is a program
  which, once evaluated, yields a value belonging to the given data type

Extending the Curry-Howard correspondence

$$t, s ::= \; x \mid \lambda x.t \mid (ts) \mid \langle t, s \rangle \mid \pi_1(t) \mid \pi_2(t) \mid \; \ldots$$

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  where t is a proof of A and s is a proof of B
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- r is a proof of $A \supset B$ iff $r = \lambda x.t$
  where $t[s/x]$ is a proof of B for any proof s of A
- $\ldots$

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- r is a proof of $A \wedge B$ iff $r \rightsquigarrow \langle t, s\rangle$
  where t is a proof of A and s is a proof of B
- r is a proof of $A \supset B$ iff $r \rightsquigarrow \lambda x.t$
  where $t[s/x]$ is a proof of B for any proof s of A
- $\dots$

The double role of BHK clauses

BHK clauses are analogous to the clauses in Tarski's truth definition

- $A \wedge B$ is true iff A is true and B is true
- $\ldots$

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Double role of Tarski's clauses (Davidson, Dummett):

- Inductively define the notion of truth;
- Specify the meaning of logically complex sentence by laying down their truth-conditions.

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Double role of BHK clauses:

- Inductively define the notion of proof;
- Specify the meaning of logically complex sentence
  by laying down their proof-conditions.

Two lines of research

- Meaning (of sentences) as proof-conditions
- Proofs as “meanings” (of formal derivations)

The meaning of derivations

An analogy

$$2 \times (4 + 3) \quad \rightsquigarrow \quad 2 \times 7 \quad \rightsquigarrow \quad 14$$

- Three expressions for the same number
- The axioms governing '+' and '×' can be viewed as
  - rewriting operations on numerical terms
  - which allow to transform any numerical term into a numeral

Natural deduction (again)

$$\frac{A \qquad B}{A \wedge B} \wedge\mathsf{I} \qquad \frac{A \wedge B}{A} \wedge\mathsf{E}_1 \qquad \frac{A \wedge B}{B} \wedge\mathsf{E}_2$$

$$\frac{\begin{matrix}[A]\\ B\end{matrix}}{A \supset B} \supset\mathsf{I} \qquad \frac{A \supset B \qquad A}{B} \supset\mathsf{E}$$

Canonical vs Non-Canonical Derivations

Closed Canonical Derivations

Directly denote proofs:

I-rules in Natural Deduction
correspond to BHK clauses

$$\dfrac{\begin{matrix}\mathscr{D}_1 \\ A\end{matrix} \qquad \begin{matrix}\mathscr{D}_2 \\ B\end{matrix}}{A \wedge B} \wedge\mathrm{I}$$

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Non-Canonical Derivations

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$$\begin{matrix}\mathscr{D} \\ A \wedge B\end{matrix} \quad \rightsquigarrow \quad \dfrac{\mathscr{D}'}{A \wedge B}\ \wedge\mathrm{I}$$

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Directly denote proofs:

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$$\dfrac{\begin{matrix}\mathscr{D}_1 \\ A\end{matrix} \qquad \begin{matrix}\mathscr{D}_2 \\ B\end{matrix}}{A \wedge B}\,\wedge\mathrm{I}$$

Non-Canonical Derivations

Indirectly denote proofs:

reduce to canonical derivations
<u>modulo</u> reduction procedures

$$\begin{matrix}\mathscr{D} \\ A \wedge B\end{matrix} \quad \rightsquigarrow \quad \dfrac{\mathscr{D}'}{A \wedge B}\,\wedge\mathrm{I}$$

Reduction procedures are the core of
Prawitz's (1965) normalization results

Inversion principle and (β-)reductions

Intro-elim patterns generate “redundancies”

Inversion principle and (β-)reductions

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$$\dfrac{\dfrac{\begin{array}{c}\overset{x}{[A]}\\ \mathscr{D}_1\\ B\end{array}}{A \supset B}\supset\mathrm{I}\ (x) \qquad \begin{array}{c}\mathscr{D}_2\\ A\end{array}}{B}\supset\mathrm{E} \qquad \rightsquigarrow_\beta \qquad \begin{array}{c}\mathscr{D}_2\\ [A]\\ \mathscr{D}_1\\ B\end{array}$$

Inversion principle and (β-)reductions

Intro-elim patterns generate "redundancies"

$$\dfrac{\dfrac{\begin{matrix}\mathscr{D}_1 \\ A_1\end{matrix} \qquad \begin{matrix}\mathscr{D}_2 \\ A_2\end{matrix}}{A_1 \wedge A_2}\wedge\mathsf{I}}{A_i}\wedge\mathsf{E}_i \qquad \rightsquigarrow_\beta \qquad \begin{matrix}\mathscr{D}_i \\ A_i\end{matrix}$$

An example

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{\overset{x}{[p \wedge q]} \qquad r}{(p \wedge q) \wedge r}\ \wedge\mathrm{I}
    }{(p \wedge q) \supset ((p \wedge q) \wedge r)}\ \supset\mathrm{I}\ (x)
    \qquad
    \dfrac{s \qquad t}{s \wedge t}\ \wedge\mathrm{I}
  }{((p \wedge q) \supset ((p \wedge q) \wedge r)) \wedge (s \wedge t)}\ \wedge\mathrm{I}
}{(p \wedge q) \supset ((p \wedge q) \wedge r)}\ \wedge\mathrm{E}
$$

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\dfrac{(p \wedge q) \supset ((p \wedge q) \wedge r) \qquad \dfrac{p \qquad q}{p \wedge q}\ \wedge\mathrm{I}}{(p \wedge q) \wedge r}\ \supset\mathrm{E}
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$$\rightsquigarrow$$

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\quad \rightsquigarrow \quad
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Normalization, confluence and fundamental fact

A normal derivation is one that is redundancy-free

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Fact. If $\mathscr{D}$ is a derivation of a complex A such that

- $\mathscr{D}$ is closed (or its active assumptions are all atomic);
- and $\mathscr{D}$ is normal.

then $\mathscr{D}$ is canonical.

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Theorem (Normalization). Every derivation in the $\{\wedge; \supset\}$-fragment of intuitionistic logic can be reduced to a normal one.

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Theorem (Church-Rosser). If $\mathscr{D} \leadsto^* \mathscr{D}_1$ and $\mathscr{D} \leadsto^* \mathscr{D}_2$ then there is a $\mathscr{D}'$ such that $\mathscr{D}_1 \leadsto^* \mathscr{D}'$ and $\mathscr{D}_2 \leadsto^* \mathscr{D}'$.

From reduction to equivalence

The reflexive, symmetric and transitive closure
of the relation induced by (β-)reductions
yields an equivalence on derivations: $\sim_\beta$

β-equivalent derivations
represent the same proof
(in different ways)

Derivations and proofs

Derivations and proofs

- Proofs are abstract entities represented/denoted by derivations
  - Normal derivations denote proofs directly
  - Non-normal derivations denote proofs indirectly

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- Proofs are abstract entities represented/denoted by derivations
  - Normal derivations denote proofs directly
  - Non-normal derivations denote proofs indirectly
- A Fregan picture:
  - Reduction changes the mode of presentation (sense)
  - but not the entity represented (denotation)

Informal vs formal characterizations of proofs

- BHK give an informal description of what proofs are
- Denotational semantics:
  - Interpret derivations as elements of a given mathematical domain
  - Derivation denoting the same proof interpreted on the same element

General Idea

Truth-Theoretic Semantics

- Truth-values
  (The True and The False)
  are the semantic values
  of sentences
- 'true' applies to a sentence A
  iff A denotes The True

General Idea

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Proof-Theoretic Semantics

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Proof-Theoretic Semantics

- Proofs
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  of what?

General Idea

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Proof-Theoretic Semantics

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- 'true' applies to a sentence A iff
  A denotes a fact

Proof-Theoretic Semantics

- Proofs
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  of valid derivations
- 'valid' applies to
  an derivation $\mathscr{D}$ for a sentence A
  iff $\mathscr{D}$ denotes a proof of A

The meaning of sentences

General Idea

Truth-Theoretic Semantics

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- 'true' applies to a sentence A iff
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Proof-Theoretic Semantics

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General Idea

Truth-Theoretic Semantics

- Facts are the semantic values of true sentences
- 'true' applies to a sentence A iff A denotes a fact

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- Proofs are the semantic values of valid derivations
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- Facts
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- ‘true’ applies to a sentence A iff
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Proof-Theoretic Semantics

- Proofs
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- ‘true’ applies to a sentence A
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General Idea

Truth-Theoretic Semantics

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Proof-Theoretic Semantics

- Proofs
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- 'true' applies to a sentence A
  iff a proof of A exists, i.e.: iff
  the set of proofs of A is non-empty

General Idea

Truth-Theoretic Semantics

- Facts
  are the semantic values
  of true sentences
- ‘true’ applies to a sentence A iff
  A denotes a fact

Proof-Theoretic Semantics

- Sets of Proofs
  are the semantic values
  of sentences
- ‘true’ applies to a sentence A
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  the set of proofs of A is non-empty

Two-valued semantics vs proof-theoretic semantics

Given the following:

Definition

A is true iff $[\![A]\!] \neq \varnothing$

A is false iff $[\![A]\!] = \varnothing$

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A more fine-grained account of meaning

$$A \dashv\vdash B$$

iff there are two derivations

$$\begin{array}{c} [B] \\ \mathscr{D}_1 \\ A \end{array} \quad \text{and} \quad \begin{array}{c} [A] \\ \mathscr{D}_2 \\ B \end{array}$$

A more fine-grained account of meaning

$$A \simeq B$$

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A more fine-grained account of meaning

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iff there are two derivations

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such that

$$[B] \sim \begin{array}{c}[B]\\ \mathscr{D}_1\\ [A]\\ \mathscr{D}_2\\ B\end{array} \qquad \text{and} \qquad \begin{array}{c}[A]\\ \mathscr{D}_2\\ [B]\\ \mathscr{D}_1\\ A\end{array} \sim [A]$$

A more fine-grained account of meaning

$$A \simeq_{?} B$$

iff there are two derivations

$$\begin{matrix} [B] \\ \mathscr{D}_1 \\ A \end{matrix} \qquad \text{and} \qquad \begin{matrix} [A] \\ \mathscr{D}_2 \\ B \end{matrix}$$

such that

$$[B] \sim_{?} \begin{matrix} [B] \\ \mathscr{D}_1 \\ [A] \\ \mathscr{D}_2 \\ B \end{matrix} \qquad \text{and} \qquad \begin{matrix} [A] \\ \mathscr{D}_2 \\ [B] \\ \mathscr{D}_1 \\ A \end{matrix} \sim_{?} [A]$$

Examples

$$A \wedge B \simeq_{\beta\eta} B \wedge A$$

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$$A \wedge A \not\simeq_{\beta\eta} A$$

Došen on formula isomorphism

That two sentences are isomorphic means that they behave exactly in the same manner in proofs: by composing, we can always extend proofs involving one of them, either as assumption or as conclusion, to proofs involving the other, so that nothing is lost, nor gained. There is always a way back. By composing further with the inverses, we return to the original proofs.

It seems reasonable to suppose that isomorphism between sentences analyzes propositional identity, i.e. identity of meaning for sentences:

> Two sentences express the same proposition if and only if they are isomorphic.

This way we would base propositional identity upon identity of proofs, since in defining isomorphism of sentences we relied essentially on a notion of identity of proofs.

Is identity of proof trivial?

Is it the case that for all derivations $\mathscr{D}_1$ and $\mathscr{D}_2$ of A:

$$\begin{array}{c}\mathscr{D}_1\\ A\end{array} \sim \begin{array}{c}\mathscr{D}_2\\ A\end{array} \qquad ?$$

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No!

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No!

$$\dfrac{\dfrac{\overset{u}{A}}{A \supset A}\,\supset\text{I (u)}}{A \supset (A \supset A)}\,\supset\text{I (w)} \qquad\qquad \dfrac{\dfrac{\overset{u}{A}}{A \supset A}\,\supset\text{I (w)}}{A \supset (A \supset A)}\,\supset\text{I (u)}$$

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$$\dfrac{\dfrac{\overset{u}{A \wedge A}}{A}\;{\wedge}\mathrm{E}_1}{(A \wedge A) \supset A}\;{\supset}\mathrm{I}\ (\mathrm{u}) \qquad\qquad \dfrac{\dfrac{\overset{u}{A \wedge A}}{A}\;{\wedge}\mathrm{E}_2}{(A \wedge A) \supset A}\;{\supset}\mathrm{I}\ (\mathrm{u})$$

Negated proposition behave classically

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On an extensional reading of functions,

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- there are only two semantic values for propositions of the form $\neg A$
- Identity of proof is trivial for such propositions
- $\neg A \dashv\vdash \neg B \quad \Rightarrow \quad \neg A \simeq \neg B \quad$ for all A, B

Concluding remarks

Thanks for your attention!