Notes on Schütte's proof of the completeness theorem for first order logic

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1 Introduction

This note contains the main steps to prove Gödel's completeness theorem for first order logic. The proof presented here can be found in full details in [1]. See also the webpage: `http://logica.uniroma3.it/~tortora/Libro.html`.

We give a mathematical form to the three words used this morning (language/proofs/semantics). This is the form given by the tradition often referred to as mathematical logic:

- language: set symbols and rules to build correct words
- proofs: derivations is a very precisely defined formal system **a proof is finite** (the set of proofs is infinite)
- semantics: models or structures=systems of values for the symbols of language (the domain of these structures is usually **infinite**): a structure realizes or not a formula (the system of values eliminates all possible choices on the meaning of a statement: once chosen the system of values the statement holds or does its negation)

2 Language

Definition 1 (Alphabet)**.** *Set of symbols that is union of the following (pairwise disjoint) sets:*

- *logical connectives ($\wedge$, $\vee$), quantifiers ($\forall$, $\exists$), constants (**T**, **F**),*
- *auxiliary symbols: (and)*
- *denumerable set $\mathcal{V} = \{v_0, v_1, \ldots\}$: first order (individual) variables*
- *set $\mathcal{C}$ of (individual) constants*
- *for $n \geq 0$ two sets $\mathcal{R}_n$ e $\mathcal{F}_n$ of predicate letters and function symbols of arity n (for $n = 0$: propositional letters and the set $\mathcal{C}$ of constants) such that:*
  - *there is a binary symmetric relation NOT on $\mathcal{R}_n$ s.t. $\mathcal{R}_n = \mathcal{R}_n^1 \dot{\cup} \mathcal{R}_n^2$ and for $i \in \{1, 2\}$, for $P \in \mathcal{R}_n^i$, there exists a unique $Q \in \mathcal{R}_n^j$ ($i \neq j$) s.t. $(P, Q) \in NOT$. Notation: $P = \neg Q$ and $Q = \neg P$.*

Definition 2 (Terms)**.** *The set $\mathcal{T}$ of terms of $\mathcal{L}$ is:*

1. *(*Induction basis*) $\mathcal{V} \cup \mathcal{C} \subseteq \mathcal{T}$;*

2. *(*Induction step*) for $n > 0$, $f \in \mathcal{F}_n$ and $t_1, ..., t_n \in \mathcal{T}$, we have $f(t_1, ..., t_n) \in \mathcal{T}$;*

3. *nothing else is a term.*

Definition 3 (Formulas)**.** *The set $\mathcal{F}$ of formulas of $\mathcal{L}$ is:*

1. *(*Induction basis*) $\mathbf{T}, \mathbf{F} \in \mathcal{F}$. For $n \geq 0$, $P \in \mathcal{R}_n$ and $t_1, ..., t_n \in \mathcal{T}$, $P(t_1, ..., t_n) \in \mathcal{F}$*

2. *(*Induction step*) For $A, B \in \mathcal{F}$ one has $(A \wedge B), (A \vee B) \in \mathcal{F}$. For $A \in \mathcal{F}$ and $x \in \mathcal{V}$ one has $\forall x A, \exists x A \in \mathcal{F}$.*

3. *nothing else is a formula.*

Definition 4 (negation)**.** 1. *$\neg(\mathbf{T}) = \mathbf{F}$ and $\neg(\mathbf{F}) = \mathbf{T}$; when $(P, Q) \in NOT$, $P, Q \in \mathcal{F}_n$ and $t_1, ..., t_n \in \mathcal{T}$, we set $\neg(P(t_1, ..., t_n)) = Q(t_1, ..., t_n)$ and thus $\neg(Q(t_1, ..., t_n)) = P(t_1, ..., t_n)$*

2. *For A, B formulas*

$$\neg(A \wedge B) = \neg(A) \vee \neg(B)$$

$$\neg(A \vee B) = \neg(A) \wedge \neg(B)$$

$$\neg(\forall x A) = \exists x \neg(A)$$

$$\neg(\exists x A) = \forall x \neg(A).$$

- examples of terms: $x + y$, $3 + x$
- examples of formulas: $x + y = x + 2$, $3 + x > z$
- negation: $\neg(\neg(P(t_1, ..., t_n))) = P(t_1, ..., t_n)$ and then for every formula A one has $\neg\neg A = A$
- $A \rightarrow B = \neg A \vee B$
- closed and open terms and formulas
- free and bounded variables
- notation: $F(x_1, \ldots, x_n)$

- substitution and equivalent formulas: substitute the term $f(x, y, z)$ to the variable v in the formula $A(v) = \forall x R(v, x)$: $\forall x R(f(x, y, z), x)$ instead of $\forall w R(f(x, y, z), w)$. So a formula is an equivalence class (up to renaming of bounded variables).

- Sequent=finite multiset of formulas

- Sequent with cyclic order= list of formulas with the natural cyclic order

3 Structures

A model is a way to "evaluate" the symbols of the language: it is a point of view on the (aparent) constants of the language. Pointer on second order quantification (example: $\forall v \forall x R(v, x)$ is not really closed, $\forall R \forall v \forall x R(v, x)$ is).

Definition 5 ($\mathcal{L}$-structures)**.** *Given a language $\mathcal{L}$, an $\mathcal{L}$-structure $\mathcal{M}$ is*

- *a set $M \neq \emptyset$ (domain of the structure, denoted also $|\mathcal{M}|$)*

- *for c an individual constant, $c_{\mathcal{M}} \in M$;*

- *for a function symbol f of arity k, $f_{\mathcal{M}} : M^k \to M$;*

- *for a propositional letter P, $P_{\mathcal{M}} \in \{0, 1\}$ and $P_{\mathcal{M}} = 1 \iff (\neg P)_{\mathcal{M}} = 0$;*

- *for a predicate letter R of arity $k \geq 1$, $R_{\mathcal{M}} \subseteq M^k$ and for $(a_1, \ldots, a_k) \in M^k$ one has $(a_1, \ldots, a_k) \in (\neg R)_{\mathcal{M}} \iff (a_1, \ldots, a_k) \notin R_{\mathcal{M}}$.*

Definition 6 (Notation: terms, formulas with parameters in a $\mathcal{L}$-structure)**.** *Given a language $\mathcal{L}$, an $\mathcal{L}$-structure $\mathcal{M}$ and $a_1, \ldots, a_n \in |\mathcal{M}|$*

- *for a term $t(x_1, \ldots, x_n, y_1, \ldots, y_m)$, we denote by $t[a_1, \ldots, a_n, y_1, \ldots, y_m]$ the ordered pair $(t(x_1, \ldots, x_n, y_1, \ldots, y_m), (a_1, \ldots, a_n))$*

- *for a formula $F(x_1, \ldots, x_n, y_1, \ldots, y_m)$, we denote by $F[a_1, \ldots, a_n, y_1, \ldots, y_m]$ the ordered pair $(F(x_1, \ldots, x_n, y_1, \ldots, y_m), (a_1, \ldots, a_n))$.*

For $t(x_1, \ldots, x_n)$ (resp. $F(x_1, \ldots, x_n)$) $t[a_1, \ldots, a_n]$ (resp. $F[a_1, \ldots, a_n]$) is closed.

Define size of a term and of a formula (with parameters): $size(A) = size(\neg A)$.

Definition 7 (value of a term/formula/sequent in a structure)**.** *$\mathcal{L}$ first order language, $\mathcal{M}$ $\mathcal{L}$-structure.*

1. *for every n, every $t(x_1, \ldots, x_n)$ and every $a_1, \ldots, a_n \in |\mathcal{M}|$, we define $t_{\mathcal{M}}[a_1, \ldots, a_n]$:*
   - *for $t = x_i$ (for some $1 \leq i \leq n$), $t_{\mathcal{M}}[a_1, \ldots, a_n] = a_i$;*
   - *for $t = c$, $t_{\mathcal{M}}[a_1, \ldots, a_n] = c_{\mathcal{M}}$;*
   - *for $t = f(t_1, \ldots, t_k)$, $t_{\mathcal{M}}[a_1, \ldots, a_n] = f_{\mathcal{M}}(t_{1_{\mathcal{M}}}[a_1, \ldots, a_n], \ldots, t_{k\mathcal{M}}[a_1, \ldots, a_n])$.*
2. *for every n, every $A(x_1, \ldots, x_n)$ and every $a_1, \ldots, a_n \in |\mathcal{M}|$, we define $\mathcal{M} \models A[a_1, \ldots, a_n]$:*
   - *$\mathcal{M} \models \mathbf{T}$ and $\mathcal{M} \not\models \mathbf{F}$;*
   - *for $A = P$ and $n = 0$, we have $\mathcal{M} \models A[a_1, \ldots, a_n] \iff P_{\mathcal{M}} = 1$;*
   - *for $A = R(t_1, \ldots, t_k)$, $\mathcal{M} \models A[a_1, \ldots, a_n] \iff (t_{1\mathcal{M}}[a_1, \ldots, a_n], \ldots, t_{k\mathcal{M}}[a_1, \ldots, a_n]) \in R_{\mathcal{M}}$;*
   - *for $A = G \wedge H$, $\mathcal{M} \models A[a_1, \ldots, a_n] \iff \mathcal{M} \models G[a_1, \ldots, a_n]$ and $\mathcal{M} \models H[a_1, \ldots, a_n]$;*
   - *for $A = G \vee H$, $\mathcal{M} \models A[a_1, \ldots, a_n] \iff \mathcal{M} \models G[a_1, \ldots, a_n]$ or $\mathcal{M} \models H[a_1, \ldots, a_n]$;*
   - *for $A = \forall v G(v, x_1, \ldots, x_n)$ (where $v \notin \{x_1, \ldots, x_n\}$), $\mathcal{M} \models A[a_1, \ldots, a_n] \iff$ for every $a \in M$ one has $\mathcal{M} \models G[a, a_1, \ldots, a_n]$;*
   - *for $A = \exists v G(v, x_1, \ldots, x_n)$ (where $v \notin \{x_1, \ldots, x_n\}$), $\mathcal{M} \models A[a_1, \ldots, a_n] \iff$ there exists $a \in M$ s.t. one has $\mathcal{M} \models G[a, a_1, \ldots, a_n]$.*

   S sequent of closed formulas: $\mathcal{M} \models S \iff$ there exists A in S and $\mathcal{M} \models A$.

- the induction is delicate
- when $n = 0$ we have the definition for closed formulas (of the language)
- $\mathcal{M} \models \neg A[a_1, \ldots, a_n] \iff \mathcal{M} \not\models A[a_1, \ldots, a_n]$
- when $\mathcal{M} \models \neg A$ (or $\mathcal{M} \not\models A$) we say that $\mathcal{M}$ is a counter model (a refutation) of A (all the point of completeness: no refutation=provable).

4 Proofs as derivations

Definition 8. *Gentzen's sequent calculus LK:*

$$\frac{}{\vdash A, \neg A}\,(\text{Ax o ID}) \qquad \frac{\vdash \Gamma, A \qquad \vdash \Delta, \neg A}{\vdash \Gamma, \Delta}\,(\text{cut})$$

$$\frac{\vdash \Gamma}{\vdash \Gamma, A}\,(W) \qquad \frac{\vdash \Gamma, A, A}{\vdash \Gamma, A}\,(C)$$

$$\frac{}{\vdash \mathbf{T}}\,(\mathbf{T}) \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \mathbf{F}}\,(\mathbf{F}) \qquad \frac{}{\vdash \Gamma, \mathbf{T}}\,(\mathbf{T})$$

$$\frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \wedge B, \Delta}\,(\wedge_m) \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}\,(\vee_m)$$

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B}\,(\vee_a^1) \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \vee B}\,(\vee_a^2) \qquad \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \ \wedge \ B}\,(\wedge_a)$$

y not free in Γ, t a term, $x_1, \ldots, x_n$ variables and $y \notin \{x_1, \ldots, x_n\}$:

$$\frac{\vdash \Gamma, A(y/x, x_1, \ldots, x_n)}{\vdash \Gamma, \forall x\, A(x, x_1, \ldots, x_n)}\,(\forall) \qquad \frac{\vdash \Delta, A(t/x, x_1, \ldots, x_n)}{\vdash \Delta, \exists x\, A(x, x_1, \ldots, x_n)}\,(\exists)$$

Sequent calculus is not the only possible way of representing proofs: here are Gentzen's rules for (some) logical connectives in natural deduction. The same rules in two sided and then one sided sequent calculus (because negation is not a connective but an involutive function). The formulation of the rules might change: no difference for provability but possibly relevant differences in the study of the behaviour of proofs

- Natural deduction (for every connective: introduction/elimination)

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \supset B, \Delta}\,\supset_i \qquad \frac{\Gamma \vdash A \supset B, \Delta \qquad \Gamma' \vdash A, \Delta'}{\Gamma, \Gamma' \vdash B, \Delta, \Delta'}\,\supset_e$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'}\,\wedge_i \qquad \frac{\Gamma \vdash A \wedge B, \Delta}{\Gamma \vdash A, \Delta}\,\wedge_e^1 \qquad \frac{\Gamma \vdash A \wedge B, \Delta}{\Gamma \vdash B, \Delta}\,\wedge_e^2$$

- Sequent calculus (for every connective: left/right introduction)

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \supset B, \Delta} \supset_r \qquad \frac{\Gamma \vdash A, \Delta \qquad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \supset B \vdash \Delta, \Delta'} \supset_l$$

$$\frac{\Gamma \vdash A, \Delta \qquad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'} \wedge_r \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge_l^1 \qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta} \wedge_l^2$$

- One sided sequent calculus (for every connective: a unique introduction)

$$\frac{\vdash \Gamma, \neg A, B}{\vdash \Gamma, A \supset B} \supset$$

$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, A \wedge B, \Delta} \wedge$$

In the sequel of this talk we will only deal with sequent calculus but Natural Deduction will be used in the sequel of the week, in particular in the lecture on the Curry-Howard correspondence.

- derivation= *finite* correct tree of rules (by induction: the set of derivable sequents). Examples:

$$\dfrac{\dfrac{\vdash \neg A, A \qquad \vdash \neg B, B}{\vdash \neg A \wedge \neg B, A, B}}{\vdash \neg A \wedge \neg B, A \vee B}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdash \neg P(z), P(z)}{\vdash \neg P(u), P(z), \neg P(z), \forall y P(y)}}{\vdash \neg P(u), P(z), \exists x(\neg P(x) \vee \forall y P(y))} \exists : x = z, \vee}{\vdash \neg P(u), \forall y P(y), \exists x(\neg P(x) \vee \forall y P(y))} \forall : z \neq u}{\vdash \exists x(\neg P(x) \vee \forall y P(y)), \exists x(\neg P(x) \vee \forall y P(y))} \exists : x = u, \vee}{\vdash \exists x(\neg P(x) \vee \forall y P(y))}$$

- analysis= (*possibly infinite*) correct tree of rules. Example of analysis that is not a derivation (bottom-up starting from $\vdash A$: W+C). By König's lemma, an analysis which is not a derivation has an infinite branch

- A (cut-free) derivable=there exists a (cut-free) derivation with conclusion A

- subformula property. A proof satisfying the subformula property is called *analytic*. For example: an analytic proof of a propositional formula will not use quantifier rules.

Theorem 1 (corectness)**.** *If the closed formula A of $\mathcal{L}$ is derivable, then, for every $\mathcal{L}$-structure $\mathcal{M}$, one has $\mathcal{M} \models A$.*

Proof. Intuitively, by induction on the derivation of A:

1. the conclusion of an axiom is satisfied by every $\mathcal{L}$-structure $\mathcal{M}$;
2. if Γ (risp. Δ) is a premise (conclusion) of a unary rule, from $\mathcal{M} \models \Gamma$ it follows that $\mathcal{M} \models \Delta$;
3. if Γ_1 and Γ_2 are the two premises of a binary rule and Δ is the conclusion, then from $\mathcal{M} \models \Gamma_1$ and $\mathcal{M} \models \Gamma_2$ it follows that $\mathcal{M} \models \Delta$.

The problem of closed formulas. □

We want to give an as much as possible constructive proof of the converse: if a formula is not provable, I can build a countermodel.

The set of rule splits: Reversible vs irreversible rules: a rule is reversible when from the derivability of the conclusion of the rule one can deduce the derivability of every premise of the rule

- the following rules are reversible: $(\mathbf{F})$, $(\wedge_a)$, $(\vee_m)$, $(\forall)$, and contracion
- the following rules are not reversible (irreversible): $(\vee_a^1)$ and $(\vee_a^2)$, $(\wedge_m)$, $(\exists)$, and weakening.

Proof. Reversibility of $\vee_m$:

$$\dfrac{\vdash \Gamma, A \vee B \qquad \dfrac{\vdash A, \neg A \qquad \vdash B, \neg B}{\vdash \neg A \wedge \neg B, A, B}}{\vdash \Gamma, A, B}$$

Irreversibility of $\vee_a$:

$$\dfrac{\vdash \mathbf{F}}{\vdash \mathbf{F} \vee \mathbf{T}}$$

There exists a $\mathcal{L}$-structure $\mathcal{M}$ which does not satisfy the premise (actually any $\mathcal{L}$-structure $\mathcal{M}$ does not satisfy $\mathbf{F}$), and thus by correctness $\mathbf{F}$ is not provable. However, $\mathbf{F} \vee \mathbf{T}$ is provable.

$\square$

Countable vs uncountable language: from now on the language is countable.

5 Where structures and proofs meet: the canonical analysis

CA is in between syntax and semantics.

We use:

- Paraproofs=proofs with hypothesis=partial proofs=finite trees of correct rules, except the leaves (not necessarily 0-ary rules of LK): a new rule is added:

$$\frac{}{\vdash \Gamma}\,(H)$$

- Para-analysis= possibly infinite paraproof (like analysis=possibly infinite derivation).

For every formula A, we want to build a para-analysis of A, called *canonical analysis* (CA), such that:

- either CA is finite and correct (thus a derivation) and A is derivable;
- or CA is finite but one 0-ary rule is not correct and from an incorrect branch we can build a countermodel of A;
- or CA is infinite and from an infinite branch we can build a countermodel of A.

5.1 Construction of the (cut-free) CA

The importance of reversible rules: we want to use only reversible rules. Two (propositional) examples:

- $X \wedge (Y \vee Z) \to ((X \wedge Y) \vee (X \wedge Z)) = (\neg X \vee (\neg Y \wedge \neg Z)) \vee ((X \wedge Y) \vee (X \wedge Z))$:

$$
\begin{prooftree}
\AxiomC{}
\doubleLine
\UnaryInfC{$\vdash \neg X, \neg Y \wedge \neg Z, X, X \wedge Z$}
\AxiomC{}
\doubleLine
\UnaryInfC{$\vdash \neg X, \neg Y \wedge \neg Z, Y, X$}
\AxiomC{}
\doubleLine
\UnaryInfC{$\vdash \neg X, \neg Y, Y, Z$}
\AxiomC{}
\doubleLine
\UnaryInfC{$\vdash \neg X, \neg Z, Y, Z$}
\BinaryInfC{$\vdash \neg X, \underline{\neg Y \wedge \neg Z}, Y, Z$}
\BinaryInfC{$\vdash \neg X, \neg Y \wedge \neg Z, Y, \underline{X \wedge Z}$}
\BinaryInfC{$\vdash \neg X, \neg Y \wedge \neg Z, \underline{X \wedge Y}, X \wedge Z$}
\UnaryInfC{$\vdash \underline{\neg X \vee (\neg Y \wedge \neg Z)}, X \wedge Y, X \wedge Z$}
\UnaryInfC{$\vdash \neg X \vee (\neg Y \wedge \neg Z), \underline{(X \wedge Y) \vee (X \wedge Z)}$}
\UnaryInfC{$\vdash \underline{(\neg X \vee (\neg Y \wedge \neg Z)) \vee ((X \wedge Y) \vee (X \wedge Z))}$}
\end{prooftree}
$$

- $(X \wedge Y) \vee (X \wedge Z)$:

$$
\begin{prooftree}
\AxiomC{}
\UnaryInfC{$\vdash \neg X, X$}
\AxiomC{}
\UnaryInfC{$\vdash \neg X, Z$}
\BinaryInfC{$\vdash \neg X, \underline{X \wedge Z}$}
\AxiomC{}
\UnaryInfC{$\vdash Y, X$}
\AxiomC{}
\UnaryInfC{$\vdash Y, Z$}
\BinaryInfC{$\vdash Y, \underline{X \wedge Z}$}
\BinaryInfC{$\vdash \underline{\neg X \wedge Y}, X \wedge Z$}
\UnaryInfC{$\vdash \underline{(\neg X \wedge Y) \vee (X \wedge Z)}$}
\end{prooftree}
$$

  If one takes $\mathcal{M} \not\models \neg X$ (i.e. $X_{\mathcal{M}} = 1$) and $\mathcal{M} \not\models Z$ (i.e. $Z_{\mathcal{M}} = 0$), one gets $\mathcal{M} \not\models \neg X, Z$, $\mathcal{M} \not\models \neg X, X \wedge Z$, $\mathcal{M} \not\models \neg X \wedge Y, X \wedge Z$, $\mathcal{M} \not\models (\neg X \wedge Y) \vee (X \wedge Z)$. Just one incorrect branch yields a countermodel: for every formula A in every sequent of the branch $\mathcal{M} \not\models A$. Since, in at least one of the leaves, no predicate P occurs with its dual $\neg P$ and $\mathbf{T}$ doesn't occur either, one can choose the value 0 for every formula in the sequent which is such a leaf.

So, we use reversible rules until possible, and if not, we nevertheless use reversible rules! More precisely, every connective/quantifier has a reversible rule (we use it), excepte $\exists$. So in that cas we proceed as follows:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(H)}
\UnaryInfC{$\vdash C(t_0/x), \ldots, C(t_n/x), \exists x C$}
\doubleLine
\RightLabel{$(\exists)$ $n+1$ times}
\UnaryInfC{$\vdash \exists x C, \ldots, \exists x C$}
\doubleLine
\RightLabel{(C) $n+1$ times}
\UnaryInfC{$\vdash \exists x C$}
\end{prooftree}
$$

This set of rules is indeed reversible: form the derivability of the conclusion one deduces the derivability of the premise (weakenings). Now the point is to show that through this method no attempt is left apart: if the procedure fails the formula was not provable. Let's do it!

What follows is not the formal description but it gives the key steps: every sequent comes equipped with a presentation (it is a *list* of formulas) and a cyclic order and a distinguished formula (we start from $\vdash A$). Go back to the examples.

Definition 9 (construction of the canonical analysis)**.** *We consider all the different cases:*

(i) In any case, when we meet a sequent of the form $\vdash A, \neg A, \Gamma$ *(or* $\vdash \Gamma, \mathbf{T}$*), we are done for this branch.*

(ii) Otherwise, we have a sequent $\vdash \Gamma, B, \Delta$ *and the construction depends on the distinguished formula* B*:*

1. *if* B *is atomic (better: no logical symbol occurs in* B*), then we just move to the next formula in the cyclic order: we substitute* B *with the next formula in* $\vdash \Gamma, B, \Delta$*. If every formula in* $\vdash \Gamma, B, \Delta$ *is atomic (this case includes the empty sequent) we have a* final hypothesis *(go back to the example).*

2. *if* $B = C \wedge D$*, we have* $\vdash \Gamma, C \wedge D, \Delta$*, and we set:*

$$\dfrac{\dfrac{}{\vdash \Gamma, C, \Delta}\,(H) \qquad \dfrac{}{\vdash \Gamma, D, \Delta}\,(H)}{\vdash \Gamma, B, \Delta}\,(\wedge_a)$$

The new distinguished formulas is the one following C *(resp.* D*) in the cyclic order.*

3. *if* $B = C \vee D$*, we have* $\vdash \Gamma, C \vee D, \Delta$*, and we set*

$$\dfrac{\dfrac{}{\vdash \Gamma, C, D, \Delta}\,(H)}{\vdash \Gamma, B, \Delta}\,(\vee_m)$$

The new distinguished formula is the one following C, D *in the cyclic order (there might be none in* Δ, Γ*, then choose one among* C, D*)*

4. *if $B = \mathbf{F}$, we have $\vdash \Gamma, \mathbf{F}, \Delta$, and we set*

$$\dfrac{\dfrac{}{\vdash \Gamma, \Delta}\,(H)}{\vdash \Gamma, B, \Delta}\,(\mathbf{F})$$

The new distinguished formula is the one following B in the cyclic order (there might be none: the empty sequent is a final hypothesis).

5. *if $B = \forall x C$, we have $\vdash \Gamma, \forall x C, \Delta$, and we set*

$$\dfrac{\dfrac{}{\vdash \Gamma, C(y/x), \Delta}\,(H)}{\vdash \Gamma, B, \Delta}\,(\forall)$$

where y is some variable that does not occur in $\vdash \Gamma, \forall x C, \Delta$. The new distinguished formula is the one following $C(y/x)$ in the cyclic order.

6. *if $B = \exists x C$, we have $\vdash \Gamma, \exists x C, \Delta$, and we set*

$$\cfrac{\cfrac{\cfrac{}{\vdash \Gamma, C(t_0/x), \ldots, C(t_n/x), B, \Delta}\,(H)}{\vdash \Gamma, B, \ldots, B, \Delta}\,(\exists)\ n+1\ \textit{volte}}{\vdash \Gamma, B, \Delta}\,(C)\ n+1\ \textit{volte}$$

*and who are $t_0, \ldots, t_n$? n is the stage of our construction (whene properly formalized, say the distance from the root of the tree) and $t_0, \ldots, t_n$ are the first $n+1$ terms in the enumeration we fixed at the beginning. We did not, we have to: so at the beginning of the whole process, we fix an enumeration of the set of terms (remember **the language is countable**).*

The new distinguished formula is the one following B in the cyclic order.

Examples: apply to the two propositional examples. And to the drunk man: to simplify take a language with no function symbols, the set of terms is $\mathcal{V}$ (the set of variables), the enumeration of terms is just the enumeration of $\mathcal{V}$ (the number of witnesses is not exactly the stage of the construction).

Set $B = \exists x(\neg P(x) \vee \forall y P(y))$

$$
\begin{array}{cl}
\hline\hline
\vdash \neg P(v_0), P(v_2), \neg P(v_0), P(v_3), \neg P(v_1), P(v_4), \neg P(v_0), \forall y P(y), \neg P(v_1), \forall y P(y), \neg P(v_2), \forall y P(y), B & \vee \\
\hline
\vdash \neg P(v_0), P(v_2), \neg P(v_0), P(v_3), \neg P(v_1), P(v_4), \neg P(v_0), \forall y P(y), \neg P(v_1), \forall y P(y), \neg P(v_2) \vee \forall y P(y), B & \\
\hline
\vdash \neg P(v_0), P(v_2), \neg P(v_0), P(v_3), \neg P(v_1), P(v_4), \neg P(v_0), \forall y P(y), \neg P(v_1) \vee \forall y P(y), \neg P(v_2) \vee \forall y P(y), B & \\
\hline
\vdash \neg P(v_0), P(v_2), \neg P(v_0), P(v_3), \neg P(v_1), P(v_4), \neg P(v_0) \vee \forall y P(y), \neg P(v_1) \vee \forall y P(y), \neg P(v_2) \vee \forall y P(y), B & \\
\hline
\vdash \neg P(v_0), P(v_2), \neg P(v_0), P(v_3), \neg P(v_1), \forall y P(y), \neg P(v_0) \vee \forall y P(y), \neg P(v_1) \vee \forall y P(y), \neg P(v_2) \vee \forall y P(y), B & \\
\hline
\vdash \neg P(v_0), P(v_2), \neg P(v_0), \forall y P(y), \neg P(v_1), \forall y P(y), \neg P(v_0) \vee \forall y P(y), \neg P(v_1) \vee \forall y P(y), \neg P(v_2) \vee \forall y P(y), & \\
\hline\hline
\end{array}
$$

$$
\begin{array}{cl}
\vdash \neg P(v_0), P(v_2), \neg P(v_0), \forall y P(y), \neg P(v_1), \forall y P(y), B & \\
\hline
\vdash \neg P(v_0), P(v_2), \neg P(v_0), \forall y P(y), \neg P(v_1) \vee \forall y P(y), B & \vee \\
\hline
\vdash \neg P(v_0), P(v_2), \neg P(v_0) \vee \forall y P(y), \neg P(v_1) \vee \forall y P(y), B & \vee \\
\hline
\vdash \neg P(v_0), \forall y P(y), \neg P(v_0) \vee \forall y P(y), \neg P(v_1) \vee \forall y P(y), B & \forall : y = v_2 \\
\hline\hline
\vdash \neg P(v_0), \forall y P(y), B & \exists : x = v_0, v_1 \\
\hline
\vdash \neg P(v_0) \vee \forall y P(y), B & \vee \\
\hline\hline
\vdash B & \exists : x = v_0
\end{array}
$$

5.2 Properties of the (cut-free) CA

Properties of CA π and of its finite approximations (stopping at some point, thus after a finite number of steps). They are para-analysis (thus trees) such that:

- the root is the sequent $\vdash A$
- the leaves are 0-ary rules of LK or (H) rules (final hypothesis for CA, final or not for approximations)
- finitely branching and every node corresponds to a rule of LK
- in the case of an approximation, a branch is a sequence of (ordered) sequents $\vdash A = \vdash \Gamma_0, \vdash \Gamma_1, \ldots, \vdash \Gamma_i$, where for $j < i$ the ordered sequent $\vdash \Gamma_j$ is conclusion of R of arity 1 or 2 and $\vdash \Gamma_{j+1}$ is one of the premises of R, and $\vdash \Gamma_i$ is conclusion of a 0-ary rule of LK or (H) (hypothesis, possibly final). If $j' \geq j$, we say $\vdash \Gamma_{j'}$ follows $\vdash \Gamma_j$ in the branch. In the case of CA its the same but the sequence might be infinite and when it is finite every hypothesis is final
- approximations and CA are cut-free: only subformulas of A occur in the tree.

- if A is provable, then every sequent occurring in CA or in any of its approximations is provable: we only used reversible rules (in the case of $\exists$ too)

- approximations of CA are finite, as a tree CA is finite or countable (countable number of nodes): at every step of the construction the approximation is finite (countable union of finite sets is finite or countable, without AC).

We call **incorrect branch** an infinite branch of CA and also a finite branch ending in a final hypothesis. **In the sequel, let** ϕ be an incorrect branch.

Lemma 1. *(i) For ϕ infinite: if C occurs in $\vdash \Gamma$ of ϕ, then for some sequent of ϕ following $\vdash \Gamma$ the formula C is the distinguished formula.*

(ii) For ϕ finite: if C occurs in $\vdash \Gamma$ of ϕ and C non atomic (better: at least one logical symbol occurs in C), then for some sequent of ϕ following $\vdash \Gamma$ the formula C is the distinguished formula.

Proof. A consequence of the fundamental remark: bottom-up, the distance from the distinguished formula to any (occurrence of) formula strictly decreases.

More precisely, for C_S occurrence of C in S with the cyclic order which is not distinguished, for every "new hypothesis" S' (with its cyclic order) and $C_{S'}$ occurence of C corresponding to C_S, the distance (in the cyclic order) from the distinguished formula of S' to $C_{S'}$ is strictly less than the distance (in the cyclic order) from the distinguished formula of S to C_S. **Check the various cases**. □

Lemma 2. *1. If $B \vee D$ occurs in $\vdash \Gamma_i$ of ϕ, then B and D occur in some $\vdash \Gamma_j$ of ϕ, with $j > i$.*

2. If $B \wedge D$ occurs in $\vdash \Gamma_i$ of ϕ, then at least one among B, D occurs in some $\vdash \Gamma_j$ of ϕ, with $j > i$.

3. If $\forall x B$ occurs in $\vdash \Gamma_i$ of ϕ, then for some variable y not occurring in C the formula $B(y/x)$ occurs in some $\vdash \Gamma_j$ of ϕ, with $j > i$.

4. If $\exists x B$ occurs in $\vdash \Gamma_i$ of ϕ, then for every term t of $\mathcal{L}$, the formula $B(t/x)$ occurs in some sequent of ϕ (and ϕ is infinite).

5. *For every atomic formula $P(x_1, \ldots, x_n)$, at most one among $P(x_1, \ldots, x_n)$ and $\neg P(x_1, \ldots, x_n)$ occurs in ϕ, and $\mathbf{T}$ does not occur in ϕ.*

Proof.

- Properties 1)2)3) follow immediately from the previous lemma
- Property 4): by the previous lemma, $\exists x B$ will be the distiguished formula infinitely many times: if you want $t = t_k$ (enumeration of terms) as witness take $\exists x B$ distiguished at distance at least k
- Property 5): the formula $P(x_1, \ldots, x_n)/\neg P(x_1, \ldots, x_n)$ occurring in a sequent of ϕ will never disappear from ϕ: thus its dual cannot occur (ϕ is incorrect). The same for $\mathbf{T}$.

□

Lemma 3. *From an incorrect branch ϕ of the CA of A, one can build a $\mathcal{L}$-strutture $\mathcal{M}$ s.t. $|\mathcal{M}|$ is the set of the closed terms of some extension of $\mathcal{L}$ and $\mathcal{M} \not\models A$.*

Proof.

- let AT =set of atomic formulas (better: no logical symbol occurs in any formula of the set) occurring in ϕ
- for every $P(x_1, \ldots, x_n)$ of $\mathcal{L}$ at most one among $P(x_1, \ldots, x_n)$ and $\neg P(x_1, \ldots, x_n)$ occurs in ϕ and $\mathbf{T}$ does not occur in ϕ (previous lemma)
- consider the extension $\mathcal{L}_\mathcal{C}$ of $\mathcal{L}$ obtained by adding a countable set $\mathcal{C}$ of constants, in bijection with the set $\mathcal{V}$ of individual variables: c_i corresponds to x_i through this bijection
- let $\neg AT_\mathcal{C}$ = set of *closed* atomic formulas of $\mathcal{L}_\mathcal{C}$ obtained from AT (not necessarily closed)
- there exists a countable $\mathcal{L}_\mathcal{C}$-structure $\mathcal{M}$ such that
  - $|\mathcal{M}|$ =set of closed terms of $\mathcal{L}_\mathcal{C}$
  - $\mathcal{M} \models \neg AT_\mathcal{C}$, where $\neg AT_\mathcal{C}$ =set of the negations of the formulas of $AT_\mathcal{C}$

  Indeed: the value of a term is the term itself and since both in $AT_\mathcal{C}$ and in $\neg AT_\mathcal{C}$ an atomic formula occurs at most once we can choose its value as we like (again, go back to the propositional example)

- we now prove that $\mathcal{M} \models \neg A$ ($\mathcal{M}$ is a $\mathcal{L}_\mathcal{C}$-structure, its restriction to $\mathcal{L}$ still satisfies $\neg A$).

More generally, we prove, by induction on $B(x_1, \dots x_n)$ of $\mathcal{L}$, that for every formula $B(x_1, \dots x_n)$ occurring in a sequent of ϕ, one has $\mathcal{M} \not\models B(c_1/x_1, \dots, c_n/x_n)$ (in particular $\mathcal{M} \not\models A$):

- if $B(x_1, \dots, x_n) = P(x_1, \dots, x_n)/\neg P(x_1, \dots, x_n)$, then $B(x_1, \dots, x_n)$ occurs in ϕ, which means that $\neg B(x_1, \dots, x_n) \in \neg AT_\mathcal{C}$ and thus $\mathcal{M} \models \neg B(c_1/x_1, \dots, c_n/x_n)$ that is $\mathcal{M} \not\models B(c_1/x_1, \dots, c_n/x_n)$

- if $B(x_1, \dots, x_n) = \mathbf{F}$, then $\mathcal{M} \not\models \mathbf{F}$, like every $\mathcal{L}_\mathcal{C}$-structure

- if $B(x_1, \dots, x_n) = C(x_1, \dots, x_n) \wedge D(x_1, \dots, x_n)$, then (by the previous lemma) either $D(x_1, \dots, x_n)$ or $C(x_1, \dots, x_n)$ occurs in ϕ, and we conclude by IH

- if $B(x_1, \dots, x_n) = C(x_1, \dots, x_n) \vee D(x_1, \dots, x_n)$, then (by the previous lemma) both $D(x_1, \dots, x_n)$ and $C(x_1, \dots, x_n)$ occur in ϕ, and we conclude by IH

- if $B(x_1, \dots, x_n) = \forall x C(x, x_1, \dots, x_n)$, then (by the previous lemma) $C(y, x_1, \dots, x_n)$ occurs in ϕ, for some variable $y = x_k$ which does not occur in $B(x_1, \dots, x_n)$. By IH $\mathcal{M} \not\models C(c_k/y, c_1/x_1, \dots, c_n/x_n)$ that is $\mathcal{M} \not\models C[c_k, c_1, \dots, c_n]$ and thus $\mathcal{M} \not\models \forall x C(x, c_1/x_1, \dots, c_n/x_n)$

- if $B(x_1, \dots, x_n) = \exists x C(x, x_1, \dots, x_n)$, then (by the previous lemma) $C(t/x, x_1, \dots, x_n)$ occurs in ϕ for every term t of $\mathcal{L}$. By IH for every closed term τ of $\mathcal{L}_\mathcal{C}$, we have $\mathcal{M} \not\models C(\tau/x, c_1/x_1, \dots, c_n/x_n)$, that is, for every $\tau \in |\mathcal{M}|$ one has $\mathcal{M} \not\models C[\tau, c_1, \dots, c_n]$: then $\mathcal{M} \not\models \exists x C(x, c_1/x_1, \dots, c_n/x_n)$.

□

5.3 The fundamental theorem

Theorem 2. *For the CA π of the closed formula A of the first order language $\mathcal{L}$, we have exactly one of the following three possibilities:*

1. *π is finite and every leaf of π is a 0-ary rule of LK (axiom or $\mathbf{T}$): then π is a cut-free derivation of A;*

2. *π is finite and there is a leaf which is a final hypothesis: then we can build an $\mathcal{L}$-strutture $\mathcal{M}$ s.t. $|\mathcal{M}|$ is the (countable) set of closed terms of a suitable extension of $\mathcal{L}$ such that $\mathcal{M} \not\models A$;*

3. *π has an infinite branch: then we can build an $\mathcal{L}$-strutture $\mathcal{M}$ s.t. $|\mathcal{M}|$ is the (countable) set of closed terms of a suitable extension of $\mathcal{L}$ such that $\mathcal{M} \not\models A$.*

Proof. If π is finite (cases 1 and 2), it is immediate. If π is infinite, it is a countable tree that by König's lemma has an infinite branch (NO CHOICE here, because we know in advance that the tree is countable). Then we can apply the previous lemma and conclude. □

5.4 Consequences of the fundamental theorem

- completeness: a closed first order formula A without countermodels is derivable in LK

  Proof. we cannot be in case 2 nor in case 3 of the fundamental theorem □

- cut-eliminability: a closed first order formula A derivable in LK is derivable without using the cut rule

  Proof. we cannot be in case 2 nor in case 3 of the fundamental theorem (using the correctness theorem) □

- subformula property: a closed first order formula A derivable in LK is derivable using only (extended) subformulas of A. A proof satisfying the subformula property is called *analytic*. For example: an analytic proof of a propositional formula will not use quantifier rules.

  Proof. immediate form the previous point □

- consistency of LK: the empty sequent is not provable in LK.

  Proof. immediate from the previous point □

5.5 Generalization of the results

- a theory T in the first order language $\mathcal{L}$ is a set of closed formulas of $\mathcal{L}$
- for T theory and A closed formula of $\mathcal{L}$, we write $T \models A$ meaning that for every $\mathcal{L}$-structure $\mathcal{M}$, if $\mathcal{M} \models T$ (i.e. $\mathcal{M} \models F$ for every $F \in T$) then $\mathcal{M} \models A$
- the formula A is derivable from the theory T (**Notation:** $\mathbf{T} \vdash \mathbf{A}$) if A is derivable in LK where we can use as 0-ary rules also the 0-ary rules with conclusion C for every $C \in T$
- generalization of the correctness theorem: for A closed formula and T theory, if A is derivable from T then $T \models A$
- generalization of the construction of the canonical analysis: the canonical analysis with cuts (CAC) of the formula A from the theory T

Theorem 3. *For the CAC π of the closed formula A from the theory T of the first order language $\mathcal{L}$, we have exactly one of the following two possibilities:*

1. *π is finite: then π is a derivation of A from T;*
2. *π has an infinite branch: then we can build an $\mathcal{L}$-strutture $\mathcal{M}$ s.t. $|\mathcal{M}|$ is the (countable) set of closed terms of a suitable extension of $\mathcal{L}$ such that $\mathcal{M} \models T \cup \{\neg A\}$.*

Consequences of the generalized theorem:

- strong completeness: if $T \cup \{\neg A\}$ is not satisfiable, then A is derivable from T

  Proof. immediate from the theorem (case 2 cannot occur) $\square$

- compactness theorem: if a theory T is finitely satisfiable (i.e. for every finite subset T_f of T there exists an $\mathcal{L}$-structure $\mathcal{M}_f$ such that $\mathcal{M}_f \models T_f$), then T is satisfiable (there exists an $\mathcal{L}$-structure $\mathcal{M}$ such that $\mathcal{M} \models T$).
  Notice that as usual in compactness in the statement there is a quantifier permutation: $\forall\exists \to \exists\forall$

Proof. if T is not satisfiable then neither $T \cup \{\neg \mathbf{F}\}$ is, then $\mathbf{F}$ is derivable from T (by the theorem), thus there exists a finite subset T_f of T such that $\mathbf{F}$ is derivable from T_f, which implies that $T_f \models \mathbf{F}$ (by correctness) that is T_f is not satisfiable $\square$

- Löwenheim-Skolem's theorem: if the theory T is satisfiable, then there exists an $\mathcal{L}$-structure with countable domain satisfying T

  Proof. if T is satisfiable, then $T \not\models \mathbf{F}$ and by correctness $\mathbf{F}$ is not derivable from T: by the fundamental theorem CAC of $\mathbf{F}$ from T is infinite and there exists an $\mathcal{L}$-strutture $\mathcal{M}$ s.t. $|\mathcal{M}|$ is the (countable) set of closed terms of a suitable extension of $\mathcal{L}$ such that $\mathcal{M} \models T(\cup\{\neg \mathbf{F}\})$. $\square$

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