Probabilistic Weakest Preconditions

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and Christoph Matheja (Lyngby)

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Overview

Probabilistic programs are hard to grasp

Does this program almost surely terminate?

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x := 1;
while (x > 0) {
    x := x+2 [1/2] x := x-1
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If not, what is its probability to diverge?

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```
x := geometric(1/4);
y := geometric(1/4);
t := x+y;
t := t+1 [5/9] skip;
r := 1;
for i in 1..3 {
   s := iid(bernouilli(1/2), 2t);
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What is the probability that r equals one on termination?

[Flajolet *et al.*, 2009]

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A powerful, simple proof calculus for probabilistic programs.

At the source code level.

No "descend" into the underlying probabilistic model.

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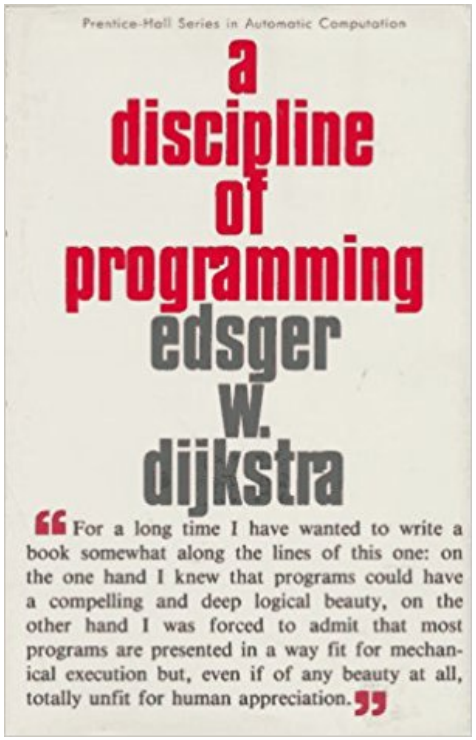
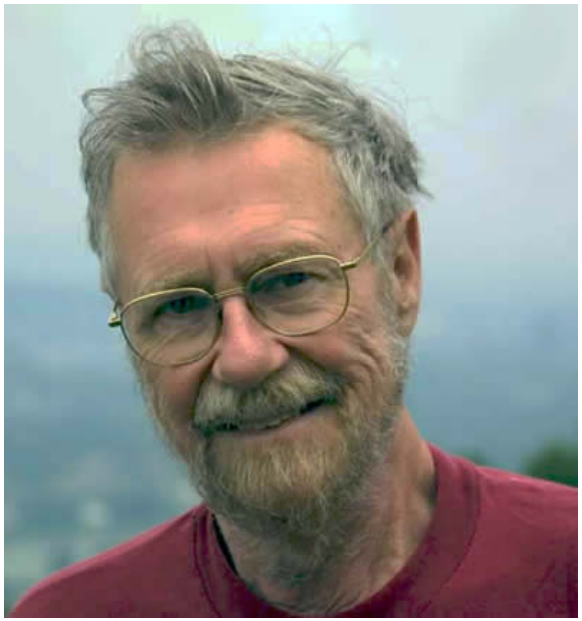
Push automation as much as we can.

Overview

A discipline of programming

WEAKEST PRECONDITIONS





Edsger Wybe Dijkstra
(1930–2002)

Predicate transformers

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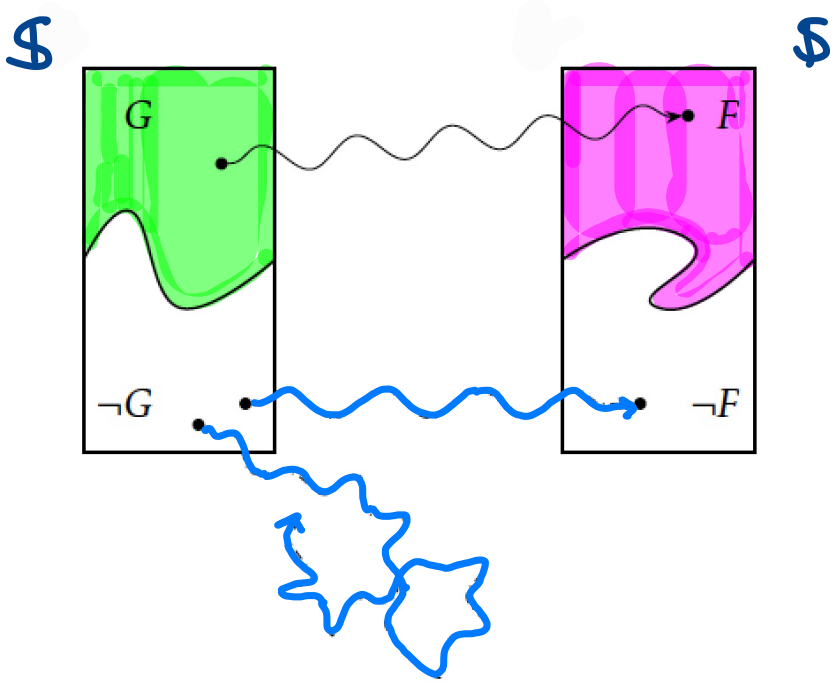

Relation to Hoare triples

Hoare logic is relational

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Weakest preconditions respect Hoare triples:

$$\{\, wp[\![P]\!](F) \,\}\ P\ \{\, F \,\} \qquad \text{is a valid statement}$$

Example

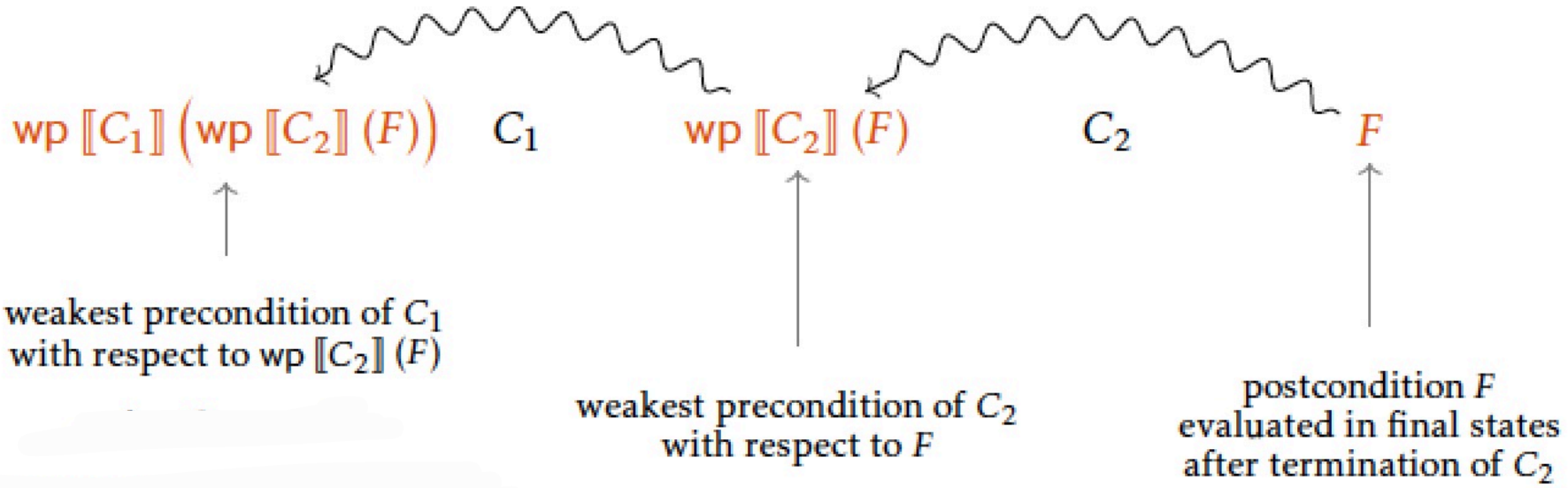
$a, b \in \mathbb{Z}$

$\{a=2 \wedge b=b\}\ a := a+2\ ;\ b := b-3\ \{a \cdot b = 12\}$

$\{a=2 \wedge b=b\}\ a := a+2\ ;\ b := b-3\ \{a=4 \vee b=a^2+1\}$

$$wp\,[\![\ a := a+2\ ;\ b := b-3\ ]\!]\,(a \cdot b = 12) = \bigvee_{a,b} (a+2)\cdot(b-3) = 12$$

Backward reasoning



Weakest preconditions reason in a backward manner about programs.

Predicate transformer semantics

Syntax program P	Semantics $wp[\![P]\!](F)$
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$x := E$	$F[x := E]$
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where lfp is the least fixed point wrt. the ordering $\sqsubseteq$ = $\Rightarrow$ on $\mathbb{P}$.

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Unbounded loops

$$wp[\![\text{while } (\varphi)\,\{\,P\,\}\,,F]\!] \;=\; \text{lfp}\,X.\;\underbrace{((\varphi \wedge wp[\![P]\!](X)) \vee (\neg\varphi \wedge F))}_{\text{loop characteristic function } \Phi_F(X)}$$

Unbounded loops

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- The function $\Phi_F : \mathbb{P} \to \mathbb{P}$ is continuous on $(\mathbb{P}, \sqsubseteq)$.
- Kleene's fixpoint theorem yields: $\text{lfp}\ \Phi_F = \sup_{n \in \mathbb{N}} \Phi_F^n(\text{false})$.
- $\Phi^n(\text{false})$ denotes the wp of running the loop exactly n times starting from $\varnothing$.

wp [[P]] (true) = set of states on which P terminates

Loops in action

while $(y>0)$ $\{y--\}$ $\qquad F = y \geqslant 0$

$$wp[\![\text{loop}]\!](y=0) = \text{lfp}\, X.\ (y>0) \wedge wp[\![y--]\!](X) \vee \neg(y>0) \wedge y \geqslant 0$$

$$= \text{lfp}\, X.\ \underbrace{\left(y>0 \wedge X[y/y-1]\right) \vee y=0}_{\Phi}$$

$$\Phi^n(\text{false}) = y=0 \vee \ldots \vee y=n-1$$

$$\sup_n \Phi^n(\text{false}) = y \in \mathbb{N}_{\geqslant 0}$$

What if programs roll dice?

Overview

"Dijkstra's weakest preconditions go random"

WEAKEST PRE-EXPECTATIONS



Dexter Kozen, Annabelle McIver, and Carroll Morgan

From predicates to quantities

- Let program P be:

  `x := 5 [4/5] x := 10`

  The expected value of `x` on P's termination is: $\frac{4}{5} \cdot 5 + \frac{1}{5} \cdot 10 = 6$

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- The probability that `x` = 10 on P''s termination is:

$$\frac{4}{5}\underbrace{\cdot[x+5 = 10]}_{\text{Iverson brackets}} + \frac{1}{5}\cdot 1 = \frac{4\cdot[x = 5] + 1}{5}$$

Expectation transformers

Let the set of expectations[1] (read: random variables):

$$\mathbb{E} = \left\{ f \mid f : \underbrace{\mathbb{S}}_{\text{states}} \to \mathbb{R}_{\geq 0} \cup \{\infty\} \right\}$$

Faggian: factors

[1] ≠ expectations in probability theory.

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expectations are the quantitative analogue of predicates

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- For P_2:: `x := x+5 [4/5] x := 10`, we have:

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If $f = [F]$ for predicate F, $wp[\![P]\!](f)$ is the probability of F on termination of P.

Kozen's duality theorem

If μ_P^s is the distribution over the final states obtained by running P on the initial state s, then for post-expectation f:

$$\underbrace{\sum_{t \in \mathbb{S}} \mu_P^s(t) \cdot f(t)}_{\text{forward}} = \underbrace{wp[\![P]\!](f)(s)}_{\text{backward}}$$

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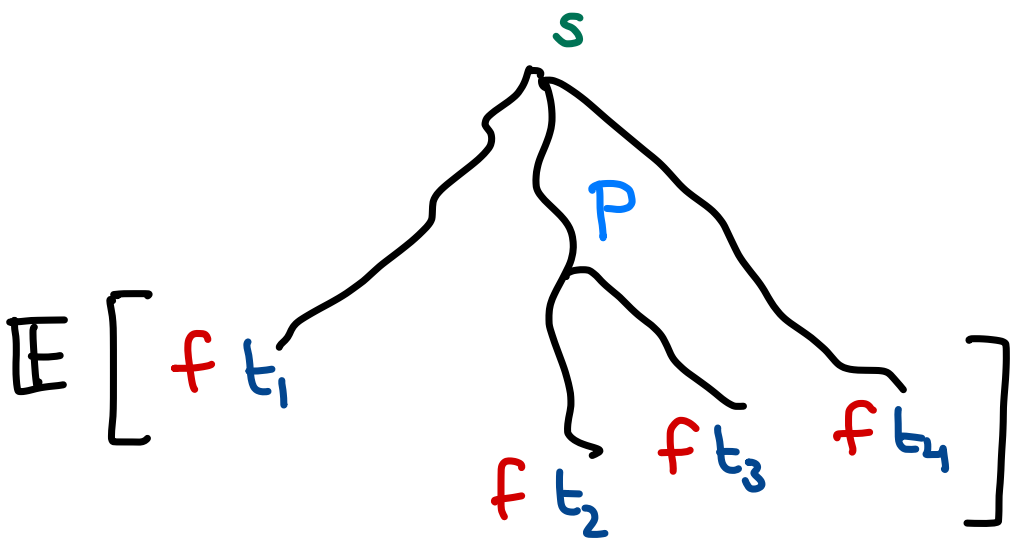
Pictorially:



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where lfp is the least fixed point wrt. the ordering $\sqsubseteq$ on $\mathbb{E}$.

Loops in action

while $(y=1)$ $\{$ $y := 0$ $[\frac{1}{3}]$ $c := c+1$ $\}$ $\qquad f = 1$

$$\Psi_f(X) = [y=1] \cdot \left(\tfrac{1}{3} X(y \leftarrow 0) + \tfrac{2}{3} X(c \leftarrow c+1) \right) + [y \neq 1] \cdot 1$$

$$\Psi_f^n(0) = [y=1] \cdot \sum_{k=0}^{n} \tfrac{1}{3} \cdot \left(\tfrac{2}{3}\right)^k + [y \neq 1]$$

$$\sup_n \Psi_f^n(0) = [y=1] \frac{1/3}{1-2/3} + [y \neq 1] = 1$$

$$= wp[\![\text{loop}]\!](1)$$

Properties

For all programs P and expectations f, g it holds:

- Continuity: $wp[\![P]\!](\cdot)$ is continuous.
- Monotonicity: $f \sqsubseteq g$ implies $wp[\![P]\!](f) \sqsubseteq wp[\![P]\!](g)$
- Feasibility: $f \sqsubseteq \mathbf{k}$ implies $wp[\![P]\!](f) \sqsubseteq \mathbf{k}$
- Linearity: $wp[\![P]\!](r \cdot f + g) = r \cdot wp[\![P]\!](f) + wp[\![P]\!](g) \qquad \forall r \in \mathbb{R}_{\geq 0}$
- Strictness: $wp[\![P]\!](\mathbf{0}) = \mathbf{0}$

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Good to know: $wp[\![P]\!](\mathbf{1})$ = termination probability of program P

Practical relevance

- Formal verification of randomised algorithms
- Exact inference for probabilistic programs
- Deciding program equivalence
- Proving program transformations
- Expected resource consumption
- Proving almost-sure termination

Overview

RELATIVE COMPLETENESS

SIAM J. COMPUT.
Vol. 7, No. 1, February 1978

SOUNDNESS AND COMPLETENESS OF AN AXIOM SYSTEM FOR PROGRAM VERIFICATION*

STEPHEN A. COOK†

Abstract. A simple ALGOL-like language is defined which includes conditional, while, and procedure call statements as well as blocks. A formal interpretive semantics and a Hoare style axiom system are given for the language. The axiom system is proved to be sound, and in a certain sense complete, relative to the interpretive semantics. The main new results are the completeness theorem, and a careful treatment of the procedure call rules for procedures with global variables in their declarations.

Key words. program verification, semantics, axiomatic semantics, interpretive semantics, consistency, completeness

1. Introduction. The axiomatic approach to program verification along the lines formulated by C. A. R. Hoare (see, for example, [6] and [7]) has received a great deal of attention in the last few years. My purpose here is to pick a simple programming language with a few basic features, give a Hoare style axiom system for the language, and then give a clean and careful justification for both the soundness and adequacy (i.e., completeness) of the axiom system. The justification is done by introducing an interpretive semantics for the language, rather like that in [10] and [8]. These two papers also have outlined soundness arguments for axiom systems, but for somewhat different language features, axioms, and interpretive models. The completeness claim and argument presented here is new (although completeness and incompleteness proofs inspired by an earlier version of this paper [2] appear in [3], [11], [12], [13], and [14]). I have tried to choose the axioms and rules of the formal system to be as simple as possible, subject to the constraints that they be sound, complete, and in the style and spirit of Hoare's rules.

SIAM J. on Computing, 1978

Stephen Cook

Two verification perspectives

- Extensional: use mathematical formulas as assertions
- Intensional: provide a syntax for assertions

Two verification perspectives

- Extensional: use mathematical formulas as assertions
- Intensional: provide a syntax for assertions
- Why bother about providing a syntax?
  - Enable automation (e.g., Dafny, Boogie, Viper, ...)
  - Often used (e.g., invariant templates, super martingales)
  - Enables guided search for specialised fragments
  - .....

Relative complete verification

Ordinary Programs

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$G \Longrightarrow wp[\![P]\!](F)$

Relative complete verification

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$F \in$ FO-Arithmetic

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modulo an oracle for deciding $\Rightarrow$

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Probabilistic Programs

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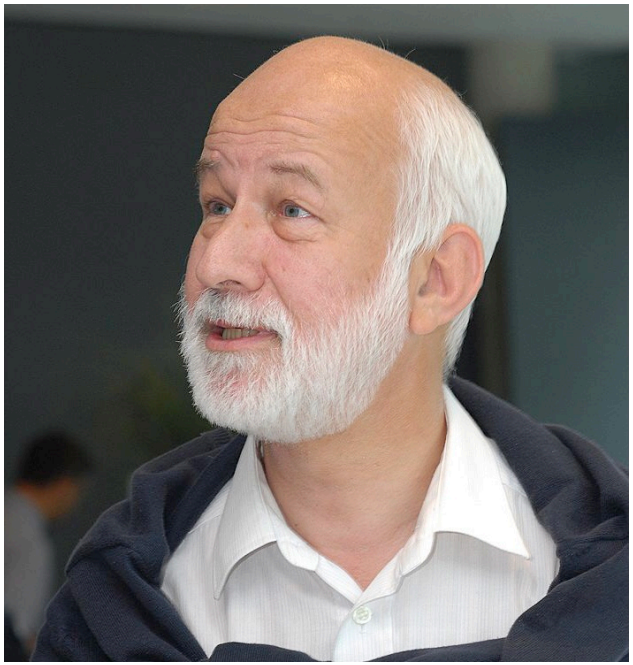
Q: How does the SomeSyntax look like?

50 years of Hoare logic

"Completeness is a subtle manner and requires a careful analysis"



Krzysztof R. Apt

Ernst-Rüdiger Olderog

Requirements on a syntax

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x := 1;
while (x > 0) {
 x +:= 2 [1/2] x -:= 1
}
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x := geometric(1/4);
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t := t+1 [5/9] skip;
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for i in 1..3 {
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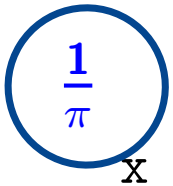


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$\mathbf{1}$

$\frac{\mathbf{1}}{\pi}$

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$[\mathbf{r = 1}]$

rational numbers, algebraic numbers, transcedental numbers, etc.

Syntax: expressions

- Arithmetic expressions

$$
\begin{array}{lcll}
a & \longrightarrow & r \in \mathbb{Q}_{\geq 0} & \text{non-negative rational} \\
 & & \mid\ x \in \mathit{Vars} & \mathbb{Q}_{\geq 0}\text{-valued variable} \\
 & & \mid\ a + a & \text{addition} \\
 & & \mid\ a \cdot a & \text{multiplication} \\
 & & \mid\ a \mathbin{\dot{-}} a & \text{subtraction truncated at zero}
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- Boolean expressions

$$
\begin{array}{lcll}
\varphi & \longrightarrow & a < a & \text{comparing arithmetic expressions} \\
 & | & \varphi \wedge \varphi & \text{conjunction} \\
 & | & \neg\varphi & \text{negation}
\end{array}
$$

Syntax: expectations

- Expectations

$$f \quad \longrightarrow \quad a \qquad \text{arithmetic expressions}$$

$$\mid \; [\varphi] \cdot f \qquad \text{guarding}$$

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- Examples:

$$\sqcup x{:}\, [x \cdot x < y] \cdot x$$

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- $f \in \mathbb{E}$ is syntactic, if f is expressible in this syntax, i.e., if $f \in \mathrm{Exp}$

Semantics of syntactic expressions

Recall that state $s : \mathit{Vars} \to \mathbb{Q}_{\geq 0}$.

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$$[\![a]\!]^s = [\![a]\!]^s$$

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$$[\![\mathcal{Q} x \colon f]\!]^s = \sup \left\{ [\![f]\!]^{s[x \mapsto r]} \mid r \in \mathbb{Q}_{\geq 0} \right\}$$

$$[\![\mathcal{U} x \colon f]\!]^s = \inf \left\{ [\![f]\!]^{s[x \mapsto r]} \mid r \in \mathbb{Q}_{\geq 0} \right\}$$

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- polynomials $y + x^3 + 2x^2 + x - 7$ widely used as templates
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$$H_x = \text{Sum}\left[v_{sum}, \frac{1}{v_{sum}}, x\right] \text{ with } [\![\text{Sum}[v_{sum}, f, x]]\!]^s = \sum_{j=0}^{s(x)} [\![f[v_{sum}/j]]\!]^s$$

Expressiveness

The set Exp of syntactic expectations is expressive

if and only if

for all programs P and $f \in \text{Exp}$ it holds

$$wp[\![P]\!]([\![f]\!]) \quad = \quad [\![g]\!]$$

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Q: is the proposed syntax expressive for loops?

A useful lemma

Any syntactic expectation $g \in \mathrm{Exp}$ can be transformed into an equivalent formula in prenex normal form:

$$Qx_1 : Qx_2 : \ldots Qx_n : f \quad \equiv \quad g$$

where $Q \in \{ \forall, \exists \}$ and f is quantifier free.

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where $Q \in \{ \mathcal{Q}, \mathcal{L} \}$ and f is quantifier free.

Remark: this does not hold if $f \cdot g$ is a first-class citizen in our syntax.
Note that $f \cdot g$ is expressible indirectly in our syntax

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Proof obligation: $\forall f \in Exp.\ \exists g \in \text{Exp}.\quad wp[\![P]\!]([\![f]\!]) = [\![g]\!]$

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Step 1: exploit Kozen's duality result to ease a syntactic characterisation

$$wp[\![P]\!]([\![f]\!]) = \lambda s_0.\ \sum_{t \in \mathbb{S}} [\![\ \underbrace{[\neg\varphi] \cdot f}_{\text{strengthen } f}\ ]\!](t) \cdot \mu_P^{s_0}(t)$$

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↑ loop body

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By Kozen's duality theorem

$$\mu^{s_i}_{P_{iter}}(\, s_{i+1} \,) \quad = \quad wp[\![P_{iter}]\!]([\![[s_{i+1}]]\!])(s_i)$$

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By Kozen's duality theorem and the induction hypothesis:

$$\mu_{P_{iter}}^{s_i}(s_{i+1}) \;=\; wp[\![P_{iter}]\!]([\![[s_{i+1}]]\!])(s_i) \overset{\text{I.H.}}{=} [\![\overbrace{g_{P_{iter}}^{s_{i+1}}}^{\text{in Exp}}]\!](s_i)$$

Wrapping it up

It follows that:

$$wp[\![\text{while } (\varphi)\{P'\}]\!](f) =$$

$$\lambda s_0.\ \mathfrak{Z}\, k\colon \sum_{s_0,\ldots,s_{k-1}} ([\neg\varphi]\cdot f)(s_{k-1}) \cdot \prod_{i=0}^{k-2} wp[\![\underbrace{\text{if } (\varphi)\{P'\} \text{ else } \{\text{skip}\}}_{\text{program } P_{iter}}]\!]([s_{i+1}])(s_i)$$

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Technical challenges in the remaining proof:

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- Express products with variable # factors in Exp (similarly)
- General products of two syntactic expectations

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It follows that:

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For details, see our POPL 2021 paper.

The expressive syntax in action

$/\!/\!/ \; \partial\, len\, \partial\, nums : \mathsf{Sum}\,[\ldots, [\mathsf{StateSeq}_x(\ldots, len)] \odot \mathsf{Path}[x](len, \ldots), nums]$ (constructive)

$= \; \boldsymbol{x + [c > 0] \cdot 2}$ (simpler syntax, equivalent expectation)

$$
\begin{array}{l}
\texttt{while}\,(c > 0)\,\{ \\
\quad \{\, c := 0 \,\}\, [1/2]\, \{\, c := 1 \,\}\,; \\
\quad x := x + 1 \\
\}
\end{array}
$$

$/\!/\!/ \; \boldsymbol{x}$

Relevance

- Relative completeness à la Cook:

bounds like $g \sqsubseteq wp[\![P]\!](f)$ are effectively decidable

[2] Deciding whether P terminates on s with *at most* probability α is Π^0_2-complete.

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- Relative completeness à la Cook:

  bounds like $g \sqsubseteq wp[\![P]\!](f)$ are effectively decidable

  algebraic, transcedental numbers via Dedekind cuts
  for $\alpha \in \mathbb{R}^{\infty}_{\geqslant 0}$: $Cut(\alpha) = \{ r \in \mathbb{Q}_{>0} \mid r < \alpha \}$

- Termination probabilities $wp[\![P]\!](\mathbf{1})$ on any input are expressible[2]

- Probability to terminate in postcondition φ as $wp[\![P]\!]([\varphi])$

- Distribution over final states where $t(x_i) = v_i$:

$$\mu_P^s(t) \quad = \quad wp[\![P]\!]\left([x_1 = v_1 \wedge \cdots \wedge x_k = v_k]\right)$$

Thus Kozen's measure transformers can be syntactically expressed

[2] Deciding whether P terminates on s with *at most* probability α is Π^0_2-complete.

Overview

Epilogue

Summary

- Weakest precondition reasoning about probabilistic programs
- $g \sqsubseteq wp[\![P]\!](f)$ is effectively decidable (modulo an oracle for $\sqsubseteq$)
- A suitable syntax for specifying quantitative "assertions"

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- A suitable syntax for specifying quantitative "assertions"

Future work

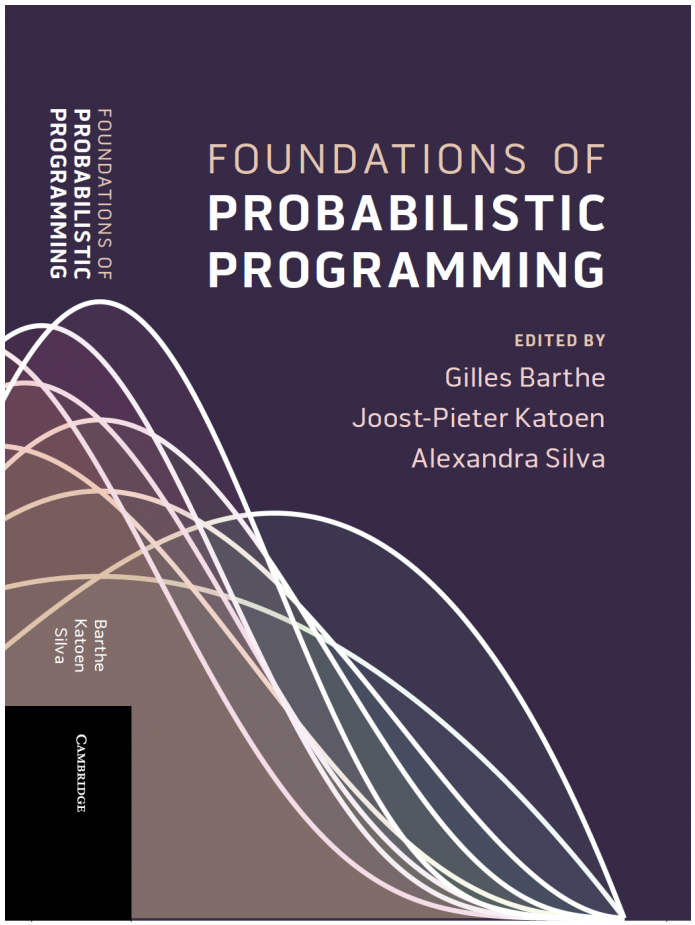
- Expressiveness of extensions (cf. next slide)
- Automation

Extensions of wp-reasoning

- . . . . . for inference (treats divergence) [TOPLAS 2018]
- . . . . . for continuous distributions w. scoring + weighting [SETTS 2019]
- . . . . . for recursion [LICS 2016]
- . . . . . for expected run-time analysis [J.ACM 2018]
- . . . . . for probabilistic separation logic [POPL 2019]
- . . . . . for weighted programs [OOPSLA 2022]

Q: which syntax suffices for these extensions?

Foundations book: open access



- semantics
- termination analysis
- concentration bounds
- algebraic reasoning
- abstract interpretation
- expected run-time analysis
- Kuifje
- Bayesian inference
- probabilistic couplings
- .....