An introduction to statistical modelling semantics with higher-order measure theory

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Logic of Probabilistic Programming
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~~Spaces~~ Statistical Modelling needs:

Discrete finite

Bool

true	false

Animal

cat
dog
Giraffe

Discrete Countable

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, String

Continuous

$\mathbb{R}^n$ $\mathbb{R}^{\mathbb{N}}$

$\mathcal{P}\mathbb{R}$,

Weight

standard Borel spaces

Measurable

Recent developments:

Discrete finite

Discrete Countable

Continuous

Probabilistic Coherence spaces & Measurable Cones [Ehrhard-Pagani-Tasson '18]

Boolean-valued Models [Bacci et al '18]

Regular ordered Banach [Dahlqvist-Kozen '20]

This talk

Quasi-Borel Spaces [Heunen et al. 17]

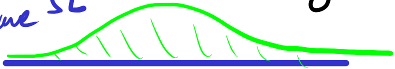
~~Measurable~~

Core idea

Measure Theory

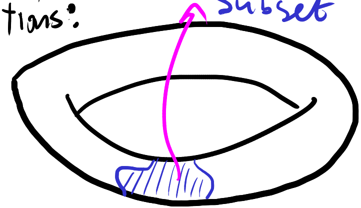
Qbs Theory

sample space Ω



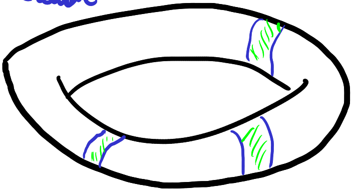
Primitive notions:

measurable subset



random element

α



Derived notions:

random elements

$\alpha : \Omega \to \text{Space}$

measure

measurable subsets

Conservative extensions:

concrete spaces we "observe"

Standard Borel spaces

Meas

Qbs

abstract spaces

(auxiliary?)

Wide topic:

Variations

Qbs, wQSs, [Vákár et al '19]

QMS, QUS,

[Forré '21]

(w)Diff, wPap [Lew et al. '22]

[Vákár et al. '20-'21]

Applications

MC inference

design &

verification [Ścibior et al '17a+b]

Network Programming

[Vandenbroucke - Schrijvers '19]

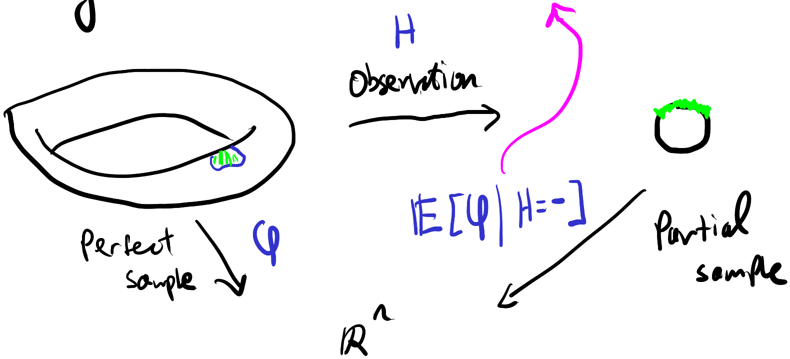
Semantics

name generation [Sabok et al. '21]

This tutorial:

- Peek behind scenes
- Gain working knowledge

Theme: higher-order measure theory demonstrated through Kolmogorov's Conditional Expectation



Kolmogorov's Conditional Expectation

- naturally higher order: $\mathbb{R}^{\Omega} \to \mathbb{R}^{\Theta}$
- behind many modern Probability techniques:
  - existence of Radon-Nikodym derivatives & density
  - existence of disintegration
  - foundation of martingales & stochastic differential equations

Agenda

I
- Borel sets
- Qbs:
  def., constructions,
  partiality, refinement
- Measures & integration

II
- Random variable spaces
- Conditional expectation

Slogan:

Measurable by Type

eg. $\{H, T\}^5$

HHTTH

$\frac{1}{32}$

Space: all possible states

Subset: all states of current interest

Measure: probability/weight/length assignment to

fine for discrete spaces

Continuous Caveat:

Thm: no $\lambda : \mathcal{P}\mathbb{R} \to [0, \infty]$:

$$\lambda(a,b) = b - a \qquad \lambda(r + A) = \lambda A \qquad \lambda\left(\biguplus_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{\infty} \lambda A_n$$

(generalises length) (translation invariant) σ-additive

Workaround: only measure well-behaved subsets

Def: The Borel subsets $\mathcal{B}_{\mathbb{R}} \subseteq \mathbb{R}$:

- open intervals $(a,b) \in \mathcal{B}_{\mathbb{R}}$

Closure under σ-algebra operations:

$$\frac{}{\emptyset \in \mathcal{B}_{\mathbb{R}}} \qquad \frac{A \in \mathcal{B}_{\mathbb{R}}}{A^c := \mathbb{R} \setminus A \in \mathcal{B}} \qquad \frac{\vec{A} \in \mathcal{B}_{\mathbb{R}}^{\mathbb{N}}}{\bigcup_{n=0}^{\infty} A_n \in \mathcal{B}_{\mathbb{R}}}$$

empty set — complements — countable unions

Examples

discrete Countable: $\{r\} = \bigcap_{\varepsilon \in \mathbb{Q}^+} (r-\varepsilon, r+\varepsilon) \in \mathcal{B}_{\mathbb{R}}$

I countable $\Rightarrow$ $I = \bigcup_{r \in I} \{r\} \in \mathcal{B}_{\mathbb{R}}$

closed intervals: $[a,b] = (a,b) \cup \{a,b\}$

Non-examples?

More complicated: analytic, Lebesgue

Def: Measurable space $V = (\llbracket V \rrbracket, B_V)$

Set (carrier)

family of subsets $B_V \subseteq P\llbracket V \rrbracket$

Closed under σ-algebra operations:

$$\frac{}{\emptyset \in B_V} \qquad \frac{A \in B_{\mathbb{R}}}{A^c := \llbracket V \rrbracket \setminus A \in B} \qquad \frac{\vec{A} \in B_{\mathbb{R}}^{\mathbb{N}}}{\bigcup_{n=0}^{\infty} A_n \in B_V}$$

empty set

complements

countable unions

Idea: structure all spaces after the worst-case scenario

Examples

- Discrete spaces $\ulcorner X^{\text{meas}} \urcorner = (X, \mathcal{P}X)$

- Euclidean spaces $\mathbb{R}^n$ — replace intervals with chests $\prod_{i=1}^{n} (a_i, b_i)$

  $\mathbb{R}^{\mathbb{N}}$ similarly

- Sub spaces: $A \in \mathcal{P}\lfloor V \rfloor$ $\quad A := (A, [B_V] \cap A)$

  $\{C \cap A \mid C \in B_V\}$

Def: Borel measurable functions $f : V_1 \to V_2$

- functions $f : \lfloor V_1 \rfloor \longrightarrow \lfloor V_2 \rfloor$
- inverse image preserves measurability:

$$f^{-1}[A] \in \mathcal{B}_{V_1} \Longleftarrow A \in \mathcal{B}_{V_2}$$

Examples

- $(+), (\cdot) : \mathbb{R}^2 \longrightarrow \mathbb{R}$
- $|-|$, $\sin : \mathbb{R} \to \mathbb{R}$
- any continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$
- any function $f : \text{`}X\text{'} \to V$

Category **Meas**

Objects: Measurable spaces

Morphisms: Measurable functions

Identities:

$$id : V \to V$$

Composition:

$$\frac{f : V_2 \to V_3 \qquad g : V_1 \to V_2}{f \circ g : V_1 \to V_3}$$

Meas Category

Products, coproducts/disjoint union, subspaces
categorical limits, colimits, but:

Thm [Aumann '61] No σ-algebra $\mathcal{B}_{\mathbb{R}^{\mathbb{R}}}$ for measurable

$$\mathrm{eval} : (\mathbf{Meas}(\mathbb{R},\mathbb{R}), \mathcal{B}_{\mathbb{R}^{\mathbb{R}}}) \times \mathbb{R} \to \mathbb{R}$$
$$(f, r) \mapsto f(r)$$

Questions? skip proof?

Proof (sketch):

Borel hierarchy:

$$
\begin{tikzcd}[column sep=small]
\Sigma^0_0 \arrow[dr, phantom, "\subsetneq" sloped] & & \Sigma^0_1 \arrow[dr, phantom, "\subsetneq" sloped] & & & & & \Sigma^0_\omega \arrow[dr, phantom, "\subsetneq" sloped] & & \\
& \Delta^0_1 \arrow[ur, phantom, "\subsetneq" sloped] \arrow[dr, phantom, "\subsetneq" sloped] & & \Delta^0_2 \arrow[ur, phantom, "\subsetneq" sloped] \arrow[dr, phantom, "\subsetneq" sloped] & & \cdots & \Delta^0_\omega \arrow[ur, phantom, "\subsetneq" sloped] \arrow[dr, phantom, "\subsetneq" sloped] & & \Delta^0_{\omega+1} \arrow[ur, phantom, "\subsetneq" sloped] \arrow[dr, phantom, "\subsetneq" sloped] & \\
\Pi^0_0 \arrow[ur, phantom, "\subsetneq" sloped] & & \Pi^0_1 \arrow[ur, phantom, "\subsetneq" sloped] & & & & & \Pi^0_\omega \arrow[ur, phantom, "\subsetneq" sloped] & &
\end{tikzcd}
$$

Stabilises at $\Delta^0_{\omega_1} = \mathcal{B}(\Sigma^0_0) = \Delta^0_{\omega_1+1}$

$$\text{rank}\ A := \min \{ \alpha < \omega_1 \mid A \in \Delta^0_\alpha \}$$

when for $B_{\mathbb{R}^\mathbb{R}} = \mathcal{P}(\mathrm{Meas}(\mathbb{R},\mathbb{R}))$

$$\mathrm{eval} : (\mathrm{Meas}(\mathbb{R},\mathbb{R}), B_{\mathbb{R}^\mathbb{R}}) \times \mathbb{R} \to \mathbb{R}$$
$$(f, r) \mapsto f(r)$$

$B_{V\times U} = B([B_V] \times [B_U])$

If measurable:

$\alpha := \sup\{\mathrm{rank}(\mathrm{eval}^{-1}[(p,q)]) \mid p, q \in \mathbb{Q}\} < \omega_1$

Take $A \in B_\mathbb{R}$, $\mathrm{rank}\,A > \alpha$

$f : \mathbb{R} \to \mathbb{R}$ $\quad f := [- \in A] := \lambda x. \begin{cases} x \in A : 1 \\ x \notin A : 0 \end{cases}$

But:

$\alpha < \mathrm{rank}\,A = \mathrm{rank}(\langle f, -\rangle^{-1}[\mathrm{eval}^{-1}[\{1\}]]) \leq \mathrm{rank}(\mathrm{eval}^{-1}[\{1\}]) \leq \alpha$

#

Sequential Higher-order structure:

I Countable : $\quad V^{\ulcorner I \urcorner} = \prod_{i \in I} V$

$\Longrightarrow$ Some higher-order structure in **Meas**:

$\mathrm{Cauchy} \in B_{[-\infty,\infty]^{\mathbb{N}}}$

$$\mathrm{Cauchy} := \bigcap_{\varepsilon \in \mathbb{Q}^+} \bigcup_{k \in \mathbb{N}} \bigcap_{\substack{m,n \in \mathbb{N} \\ m,n \geq k}} \{ \vec{y} \in [-\infty,\infty]^{\mathbb{N}} \mid |y_m - y_n| < \varepsilon \}$$

$\limsup : [-\infty,\infty]^{\mathbb{N}} \to [-\infty,\infty]$ $\quad$ $\lim : \mathrm{Cauchy} \to \mathbb{R}$

Compose higher-order building blocks:

lim is measurable!

$$\mathrm{VanishingSeq}(\mathbb{R}) := \left\{ \vec{r} \in \mathbb{R}^{\mathbb{N}} \;\middle|\; \lim_{n\to\infty} r_n = 0 \right\} \in B_{\mathbb{R}^{\mathbb{N}}}$$

$$\mathrm{approx}\_ : \mathrm{VanishingSeq}(\mathbb{R}^{+}) \times \mathbb{R} \longrightarrow \mathbb{Q}^{\mathbb{N}}$$

$$\text{s.t.:} \quad \left| \left(\mathrm{approx}_{\vec{\Delta}}\, r\right)_n - r \right| < \Delta_n$$

Slogan: Measurable by Type !

Not all operations of interest fit:

Intrinsically higher-order !

$$\limsup : \left([-\infty,\infty]^{\mathbb{R}}\right)^{\mathbb{N}} \longrightarrow [-\infty,\infty]^{\mathbb{R}}$$

$$\limsup := \lambda \vec{f}.\lambda x.\, \limsup_{n\to\infty} f_n\, x$$

Want

Slogan: Measurable by Type!

But

For higher-order building-blocks, must

defer measurability proofs until we're

1st order again. ⇒ non-compositionality

Plan

Def: $V \in \mathbf{Meas}$ is Standard Borel when

$$V \cong A \quad \text{for some } A \in \mathcal{B}_{\mathbb{R}}$$

the "good part" of **Meas** — the subcategory

$$\mathbf{Sbs} \hookrightarrow \mathbf{Meas}$$

Sbs include

- Discrete $\ulcorner I \urcorner$, I countable
- Countable products of Sbs:

$$\mathbb{R}^n, \mathbb{R}^{\mathbb{N}}, 2^{\mathbb{N}}, \mathbb{N}^{\mathbb{N}}$$

- Borel subspaces of Sbs:

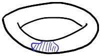
$\mathbb{I} := [0,1]$ $\mathbb{R}^+ := (0,\infty)$ $\mathbb{R}_{\geq 0} := [0,\infty]$

- Countable coproducts of Sbs:

$\mathbb{W} := [0,\infty]$ $\overline{\mathbb{R}} := [-\infty,\infty]$

Agenda

Slogan: Measurable by Type

- Borel sets 
- Qbs:
  def., constructions,
  partiality, refinement
- Measures & integration
- Random variable spaces
- Conditional expectation

Def: Quasi-Borel space $X = (\lfloor X \rfloor, R_X)$

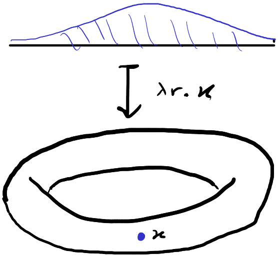
set "carrier"

set of functions $\alpha : \mathbb{R} \to \lfloor X \rfloor$ "random elements"

$R_X \subseteq \lfloor X \rfloor^{\lfloor \mathbb{R} \rfloor}$ closed under:

- constants:

$$\frac{x \in \lfloor X \rfloor}{(\lambda r. x) \in R_X}$$



- precomposition:

- recombination

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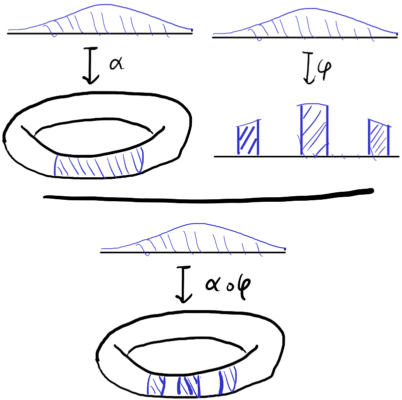
$R_X \subseteq \lfloor X \rfloor^{\lfloor \mathbb{R} \rfloor}$ closed under:

- precomposition:

$$\frac{\alpha \in R_X \qquad \varphi : \mathbb{R} \to \mathbb{R} \text{ in Sbs}}{\varphi \circ \alpha : \mathbb{R} \xrightarrow{\varphi} \mathbb{R} \xrightarrow{\alpha} \lfloor X \rfloor \in R_X}$$



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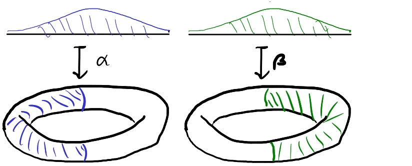
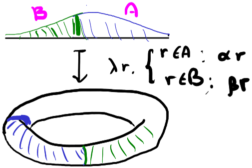
- recombination

$$\frac{\vec{\alpha} \in R_X^{\mathbb{N}} \qquad \mathbb{R} = \biguplus_{n=0}^{\infty} \underbrace{A_n}_{\in B_{\mathbb{R}}}}{\lambda r. \begin{cases} \vdots \\ r \in A_n : \alpha_n r \\ \vdots \end{cases}}$$





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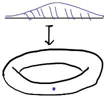
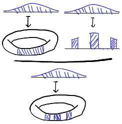
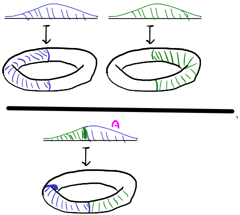
- constants: 

- precomposition: 

- recombination 

Examples

recombination of constants

- $\mathbb{R} = (\lfloor \mathbb{R} \rfloor, \mathrm{Meas}(\mathbb{R}, \mathbb{R}))$

  qbs underlying $\mathbb{R}$

  $\lambda r. \left\{ \begin{array}{l} \vdots \\ r \in A_n : x_n \\ \vdots \end{array} \right.$

- $X \in \mathrm{Set}$, $\ulcorner X \urcorner^{\mathrm{Qbs}} := (X, \sigma\text{-}\mathrm{Simple}(\mathbb{R}, X))$

  discrete qbs on X

- " $\quad \llcorner X \lrcorner_{\mathrm{Qbs}} := (X, X^{\lfloor \mathbb{R} \rfloor})$

  all funtions

  Indiscrete qbs on X

Qbs morphism $f : X \to Y$

- function $f : \lfloor X \rfloor \to \lfloor Y \rfloor$
- $$\frac{\begin{array}{c}\mathbb{R} \\ \alpha \downarrow \\ \lfloor X \rfloor\end{array} \in R_X}{\begin{array}{c}\mathbb{R} \\ \alpha \downarrow \\ \lfloor X \rfloor \\ f \downarrow \\ \lfloor Y \rfloor\end{array} \in R_Y}$$

Example

- Constant functions are qbs morphisms
- σ-simple functions are qbs morphisms

Category Qbs $\Leftarrow$ - identity, composition

Useful adjunctions:

$$\begin{tikzcd}
\mathbf{Meas} \arrow[d, "\lfloor \overline{-} \rfloor_{\mathrm{Qbs}}"] \\
\mathbf{Qbs} \arrow[u, bend left, "\lceil - \rceil^{\mathrm{Meas}}"] \arrow[d, "\lfloor - \rfloor"] \\
\mathbf{Set} \arrow[u, bend left, "\lceil - \rceil^{\mathrm{Qbs}}"] \arrow[u, bend right, "\lfloor \overline{\mathrm{Qbs}} \rfloor"']
\end{tikzcd}$$

$\lfloor V \rfloor_{\mathrm{Qbs}} := (\lfloor V \rfloor, \mathbf{Meas}(\mathbb{R}, V))$

$(V \in \mathbf{Meas})$

$\lceil X \rceil^{\mathrm{Meas}} := \left\{ A \subseteq \lfloor X \rfloor \;\middle|\; \forall \alpha \in R_X .\ \alpha^{-1}[A] \in \mathcal{B}_{\mathbb{R}} \right\}$

- limits (products, subspaces) and colimits (coproducts, quotients) as in **Set**
- Slogan: every measurable space is carried by a qbs

Example

Product $(X \times Y, \pi_1, \pi_2)$:

necessarily!

- $\lfloor X \times Y \rfloor = \lfloor X \rfloor \times \lfloor Y \rfloor$

- $R_{X \times Y} = \{ \lambda r. (\alpha r, \beta r) \mid \alpha \in R_X, \beta \in R_Y \}$

correlated random elements

rest of structure as in **Set**.

Function Spaces

Straightforward!

- $\lfloor Y^X \rfloor := \mathbf{Qbs}(X, Y)$
- $R_{Y^X} := \mathrm{uncurry}[\mathbf{Qbs}(\mathbb{R} \times X, Y)]$

$$= \left\{ \alpha : \mathbb{R} \to \lfloor Y^X \rfloor \;\middle|\; \lambda(r,x).\, \alpha\, r\, x : \mathbb{R} \times X \to Y \right\}$$

- $\mathrm{eval} : Y^X \times X \to Y$

$$\mathrm{eval}(f, x) := f\, x$$

Meas vs Qbs

By generalities:

$$\begin{tikzcd} \mathbf{Meas} \arrow[d, bend left, "L_{\mathrm{Qbs}}"] \arrow[d, phantom, "\dashv"] \\ \mathbf{Qbs} \arrow[u, bend left, "\ulcorner - \urcorner^{\mathrm{Meas}}"] \end{tikzcd}$$

σ-algebra on $\mathbf{Meas}(\mathbb{R},\mathbb{R})$

$$\begin{tikzcd} \ulcorner \mathbb{R}^{\mathbb{R}}_{\mathrm{Meas}} \urcorner \times \mathbb{R} \arrow[r] \arrow[rr, bend right, "\ulcorner \mathrm{eval}^{\mathrm{Meas}} \urcorner"'] & \ulcorner \mathbb{R}^{\mathbb{R}} \urcorner^{\mathrm{Meas}} \times \mathbb{R} \arrow[r, dashed, "\times"] & \ulcorner \mathbb{R} \urcorner = \mathbb{R} \end{tikzcd}$$

No factorisation by Aumann's Theorem.

$$\left(\underline{\text{So}} \quad \ulcorner \mathbb{R}^{\mathbb{R}} \times \mathbb{R} \urcorner \not\cong \ulcorner \mathbb{R}^{\mathbb{R}} \urcorner \times \ulcorner \mathbb{R} \urcorner \right)$$

Random element Space

$$R_X := X^{\mathbb{R}} \quad \text{since} \quad \lfloor X^{\mathbb{R}} \rfloor = R_X \text{ as sets.}$$

Why?

$(\subseteq)$ $\alpha \in \lfloor X \rfloor^{\mathbb{R}} \Rightarrow \alpha : \mathbb{R} \to X$ in Qbs.

$\mathrm{id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ measurable $\Rightarrow \mathrm{id} \in R_{\mathbb{R}}$

$\Rightarrow \alpha = \alpha \circ \mathrm{id} \in R_X$

Precomposition

$(\supseteq)$ $\alpha \in R_X \Rightarrow \forall \varphi \in R_{\mathbb{R}} = \mathrm{Meas}(\mathbb{R}, \mathbb{R}).\ \alpha \circ \varphi \in R_X \Rightarrow \alpha : \mathbb{R} \to X$

$\Rightarrow \alpha \in \lfloor X^{\mathbb{R}} \rfloor$

Subspaces

For $X \in \mathbf{Qbs}$, $A \subseteq \lfloor X \rfloor$ set:

$$R_A := \{\alpha : \mathbb{R} \to A \mid \alpha \in R_X\}$$

Then $A = (A, R_A)$ is the subspace qbs

We write $A \hookrightarrow X$

Borel subspaces ensemble

The σ-algebra $B_X := \{ A \subseteq \lfloor X \rfloor \mid \forall \alpha \in R_X . \alpha^{-1}[A] \in B_{\mathbb{R}} \}$

internalises as $B_X = 2^X$, the qbs of

Borel subsets.

($\lfloor B_{(B_{\mathbb{R}})} \rfloor$ are the Borel-on-Borel sets from descriptive set theory.
cf. [Saban et al. '21])

Standard Borel spaces

Def: A qbs S is standard Borel when

$$S \cong A \quad \text{for some } A \in \mathcal{B}_{\mathbb{R}}$$

$$\begin{tikzcd}
\mathbf{Meas} \arrow[r, bend left] \arrow[r, phantom, "\top"] & \mathbf{Qbs} \arrow[l, bend left] \\
\mathbf{Sbs} \arrow[u, hook] \arrow[r, bend left] \arrow[r, phantom, "\cong"] & \mathbf{Sbs} \arrow[u, hook] \arrow[l, bend left]
\end{tikzcd}$$

Slogan: Qbs conservative extension of Sbs

Example

$C_0 := \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ continuous} \} \hookrightarrow \mathbb{R}^{\mathbb{R}}$

C_0 is sbs. (Well-known!)

Proof:

$C_0' \in B_{\mathbb{R}^{\mathbb{Q}}}$ sbs!

$$C_0' := \left\{ g \in \mathbb{R}^{\mathbb{Q}} \;\middle|\; \begin{array}{l} \forall a, b \in \mathbb{Q}, \varepsilon \in \mathbb{Q}^+ \\ \exists \delta \in \mathbb{Q}^+ \; \forall p, q \in \mathbb{Q}^+ \cap [a,b]. \\ |p - q| < \delta \Rightarrow |g\,p - g\,q| < \varepsilon \end{array} \right\}$$

Borel measurable by type checks

on closed intervals (= compact intervals)

Continuity $\Updownarrow$ uniform continuity

then $C_0 \cong C_0' \in B_{\mathbb{R}^{\mathbb{Q}}}$:

$$\begin{array}{ll} C_0 \to C_0' & C_0' \to C_0 \\ \varphi \mapsto \varphi|_{\mathbb{Q}} & \varphi \mapsto \lambda r. \lim\limits_{n \to \infty} g(\text{approx}_{(\frac{1}{m})_{m \in \mathbb{N}}} r)_n \end{array}$$

Example (ctd)

C_0 is sbs, and $\mathrm{eval}: C_0 \times \mathbb{R} \to \mathbb{R}$

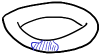
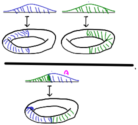
is a measurable.

Avoids:

- constructing complete separable metrics
- proving that evaluation is measurable w.r.t. metric σ-algebra.

Agenda

Slogan: Measurable by Type

- Borel sets 
- Qbs: 
  def., constructions,
  partiality, refinement
- Measures & integration
- Random variable spaces
- Conditional expectation

Partiality cf. [Vákár et al.'19]

A Borel embedding $e : X \hookrightarrow Y$

- injective function $e : \lfloor X \rfloor \rightarrowtail \lfloor Y \rfloor$
- its image is Borel: $e[\lfloor X \rfloor] \in B_Y$
- e is Strong: $\alpha \in R_X \iff e \circ \alpha \in R_Y$

Examples

- $\mathbb{1} \hookrightarrow \mathbb{2}$
- S is sbs $\Leftrightarrow \exists\, S \hookrightarrow \mathbb{R}$

Non-examples $\rightsquigarrow$ [Sabok et al. '21]

- $\{ A \in B_{\mathbb{R}} \mid A \neq \emptyset \} \hookrightarrow B_{\mathbb{R}}$

- $\{ (A_1, A_2) \in B^2_{\mathbb{R}} \mid A \subseteq B \} \hookrightarrow B^2_{\mathbb{R}}$

- $\{ A \in B_{\mathbb{R}} \mid A \text{ open} \} \hookrightarrow B_{\mathbb{R}}$

Def: A Partial map $f : X \rightharpoonup Y$ is a morphism

$$f : X \to Y \amalg \{\bot\}$$

Its domain of definition $\mathrm{Dom}\, f := \{x \mid fx \neq \bot\}$ $\hookrightarrow X$

Partial hom-sets are ordered:

for $f, g : X \rightharpoonup Y$ $\quad f \leq g$ when $\forall x.\ fx \neq \bot \Rightarrow gx = fx.$

[Cockett-Lack '06]

A model of restriction categories / axiomatic domain theory

[Fiore_Plotkin '94] Borel embeddings are the admissible monos

Space refinement

Let $P : X \to (\lfloor 2 \rfloor_{qbs})^Y$ X-parametrised property

indiscrete 2-elmt qbs

$\prod_{x \in X} P_x \hookrightarrow Y^X$ $\quad\quad$ $\coprod_{x \in X} P_x \hookrightarrow X \times Y$

$\prod_{x \in X} P_x := \{ f \in Y^X \mid \forall x \in X. f x \in P_x \}$ $\quad\quad$ $\coprod_{x \in X} P_x := \{ (x, y) \mid y \in P_x \}$

When P factors as $P : X \xrightarrow{Q} 2^Y \xrightarrow{\lambda U.U} \lfloor 2 \rfloor_{qbs}^Y$,

Write $\prod_{x \in X} Q_x$ $\quad$ $\coprod_{x \in X} Q_x$ for the same spaces

Example $(\Omega \in Qbs)$

Converging $\hookrightarrow ([-\infty,\infty]^{\Omega})^{\mathbb{N}} \cong ([-\infty,\infty]^{\mathbb{N}})^{\Omega}$

$$\text{Converging} \cong \prod_{\omega\in\Omega} \{\vec{f} \mid \exists \lim_{n\to\infty} f_n\omega\}$$

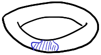
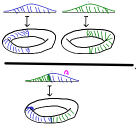
Refinement not dependent types

$\prod_X P$ require all $P_x \hookrightarrow Y^X \rightsquigarrow$ independently of x

Qbs can interpret dependent types, but such ensemble spaces require a to-be-determined universe.

Agenda

Slogan: Measurable by Type

- Borel sets 
- Qbs:
  def., constructions, 
  partiality, refinement $\to$, $\Rightarrow$, $\coprod$, $\prod$
- Measures & integration
- Random variable spaces
- Conditional expectation

Def: A measure μ over $\mathbb{R}$ is a function

$$\mu : B_{\mathbb{R}} \to W := [0,\infty]$$

s.t. - $\mu\emptyset = 0$

$$- \quad \frac{\vec{A} \in B_{\mathbb{R}}^{\mathbb{N}} \qquad \begin{array}{c} A_n \cap A_m = \emptyset \\ (n \neq m) \end{array}}{\mu\left(\bigcup_{n=0}^{\infty} A_n\right) = \sum_{n=0}^{\infty} \mu A_n}$$

For measurable spaces, replace $\mathbb{R}$ with V

We write $\lfloor G V \rfloor$ for the set of measures on V

For qbs X, take $\lfloor G \ulcorner \underset{X}{\mathrm{Meas}} \urcorner \rfloor$

The unrestricted Giry space

Equip $\lfloor GV \rfloor$ with

$$R_{GV} := \{ \alpha : \mathbb{R} \to GV \mid \forall A \in \mathcal{B}_V .\ \lambda r.\ \alpha(r, A) : \mathbb{R} \to \mathbb{W} \}$$

α is a kernel.

Farewell Meas

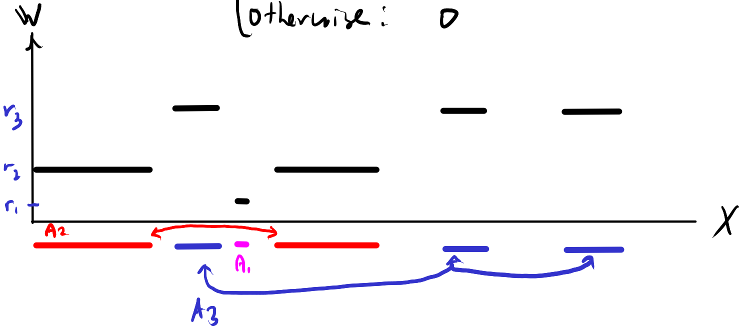
Now on: "measurable function" means

qbs morphism!

Def: Simple function $\varphi : X \to W$ when

$\exists\, n \in \mathbb{N},\ \vec{A} \in B_X^n,\ A_i \cap A_j = \emptyset\ (i \neq j),\ \vec{r} \in W$ s.t.

$$\varphi x = \begin{cases} \vdots & \\ x \in A_i : & r_i \\ \vdots & \\ \text{otherwise} : & 0 \end{cases}$$



Encode into a space:

$$\mathrm{SimpleCode} := \coprod_{n \in \mathbb{N}} B_X^n \times W^n$$

$$\mathrm{Simple} := \{ f \in W^X \mid f \text{ simple} \} \hookrightarrow W^X$$

and define an interpretation:

$$[\![-]\!] : \mathrm{SimpleCode} \twoheadrightarrow \mathrm{Simple}$$

$$[\![(n, \vec{A}, \vec{r})]\!] := \sum_{i=1}^{n} r_i \cdot [- \in A_i]$$

↳ characteristic function for A_i

Lemma: $f : X \to W$ is measurable iff $f = \lim_{n\to\infty} f_n$ for some monotone sequence $\vec{f} \in \mathrm{Simple}$.

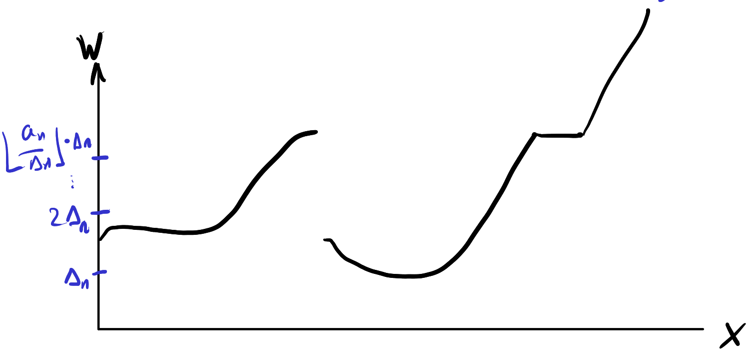
remember! ωbs morphism!

Moreover, we have measurable such choice:

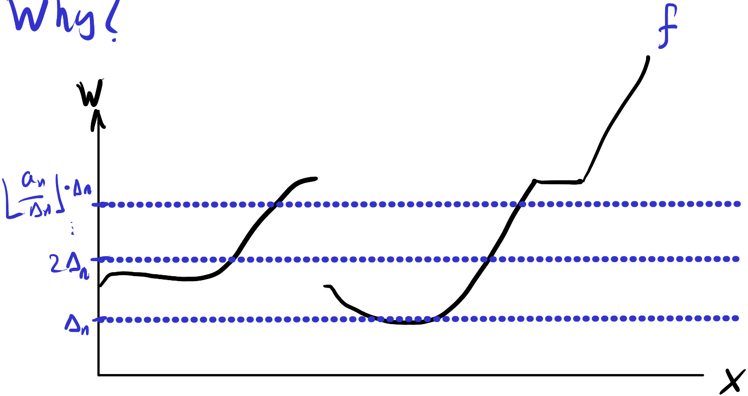
Simple Approx :

$$\{\vec{\Delta} \in \mathbb{R}^+ \mid \Delta_n \to 0\} \times \{\vec{a} \in W^{\mathbb{N}} \mid \begin{array}{c}\vec{a} \text{ monotone} \\ a_n \to \infty\end{array}\} \times W^X \to \mathrm{SimpleCode}$$

Why?


f
W
X
$\lfloor \frac{a_n}{\Delta_n} \rfloor \cdot \Delta_n$
$2\Delta_n$
Δ_n

Why?



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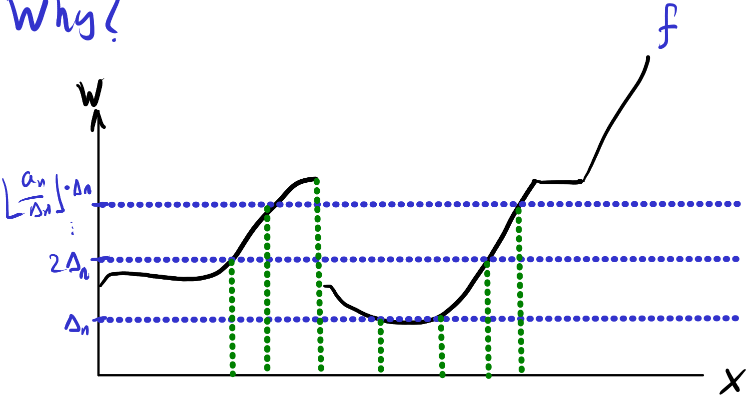

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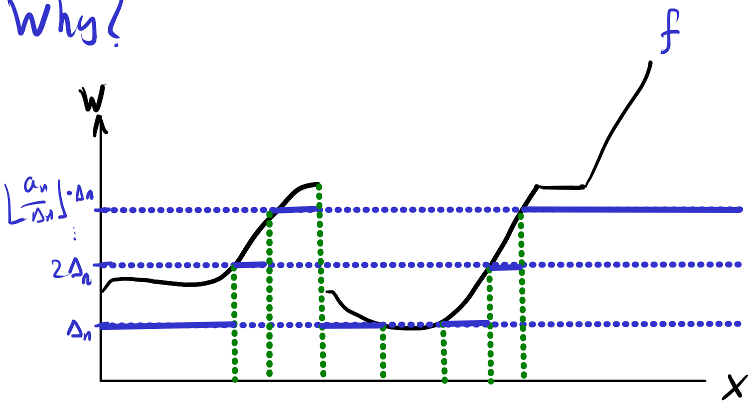

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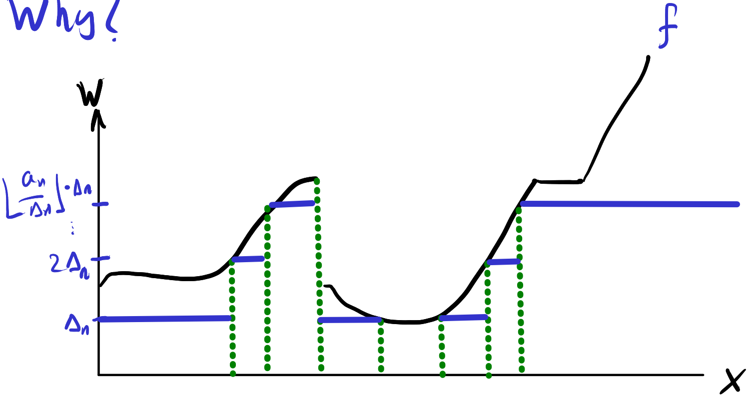

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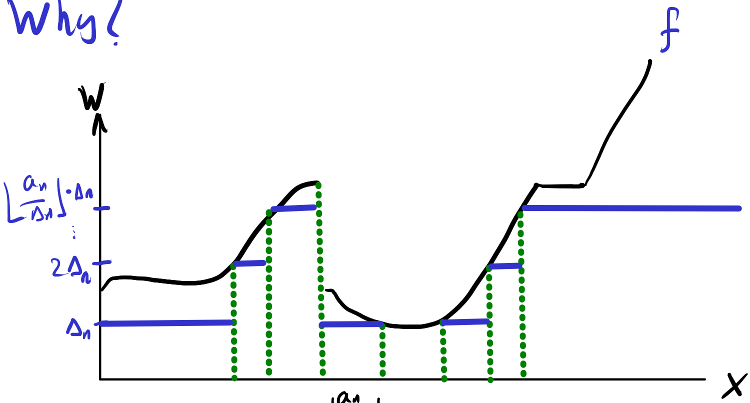
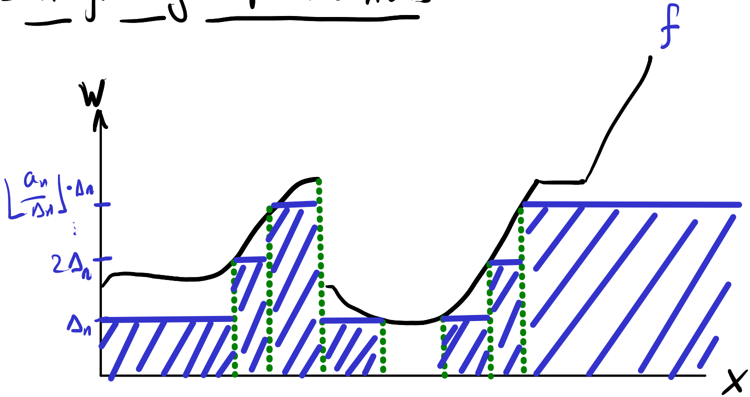
Why?



$$[\![\mathrm{SimpleApprox}_{\vec{\Delta},\vec{a}}\, f]\!] := \sum_{i=1}^{\lfloor \frac{a_n}{\Delta_n} \rfloor} i \cdot \Delta_n \left[i \cdot \Delta_n \leq f < (i+1)\Delta_n \right]$$

$$+ \left\lfloor \frac{a_n}{\Delta_n} \right\rfloor \Delta_n \cdot \left[f \geq \left\lfloor \frac{a_n}{\Delta_n} \right\rfloor \cdot \Delta_n \right] \in \mathrm{Simple}$$

Integrating Simple Functions



$$\int : GX \times \mathrm{SimpleCode} \to W$$

$$\int \mu\,(n, \vec{A}, \vec{r}) := \sum_{I \subseteq \{1,\ldots,n\}} \Big(\sum_{i \in I} r_i\Big) \cdot \mu\Big(\bigcap_{i \in I} A_i \setminus \bigcup_{i \notin I} A_i\Big)$$

Integration

Properly higher-order operation

$$\int : GX \times W^X \longrightarrow W$$

$$\int \mu f := \sup\left\{ \int \mu \varphi \;\middle|\; \varphi \in \mathrm{Simple},\ \varphi \le f \right\}$$

$$= \lim_{n\to\infty} \int \mu (\mathrm{SimpleApprox}_{\vec{\Delta}, \vec{a}}\, f)_n$$

measurable by type

we also write $\int \mu(dx)t$ for $\int \mu(\lambda x; t)$

for $\frac{a_n}{\Delta_n} \to 0$, eg. $\Delta_n = \frac{1}{2^n}$ $a_n = n$.

The unrestricted Giry Strong Monad

Unlike the unrestricted Giry on Meas.

Dirac:

$$\delta : X \to GX$$

$$x \mapsto \lambda A. \begin{cases} x \in A : & 1 \\ x \notin A : & 0 \end{cases}$$

but: non-commutative

(Fubini fails, just like in Meas)

Kleisli extension/Kock integral:

$$\oint : GX \times GY^X \longrightarrow GY$$

$$\oint \mu f := \lambda A. \int \mu(dx)\, f\, x\, A$$

Randomisable measures monad

$D \rightarrowtail G$

$\lambda A. \int_{\mathrm{Dom}\,\alpha} \lambda \; \alpha^{-1}[A]$

$$\lfloor DX \rfloor := \{ \alpha_* \lambda \mid \alpha : \mathbb{R} \to X \}$$

Lebesgue measure

$$M_{DX} := \{ \lambda x. (\alpha x)_* \lambda \mid \alpha : \mathbb{R} \times \mathbb{R} \to X \}$$

D is commutative (Fubini's Theorem)

$\mu \in DX$, $\nu \in DY$:

$$\oint \mu(dx) \oint \nu(dy)\, \delta_{(x,y)} = \oint \nu(dy) \oint \mu(dx)\, \delta_{(x,y)} =: \mu \otimes \nu$$

Models Kock's Synthetic measure theory [Kock'12, Scibior et al. 18]

Distribution Submonads

A **measure space** $\Omega = (\Omega, \mu)$ is a qbs Ω with $\mu \in DX$.

Similarly: – finite measure space
– (sub) probability space.

$$
\begin{array}{c}
PX := \{\mu \in DX \mid \mu X = 1\} \\
\downarrow \\
P_{\leq 1} X := \{\mu \in DX \mid \mu X \leq 1\} \\
\downarrow \\
P_{<\infty} X := \{\mu \in DX \mid \mu X < \infty\} \\
\downarrow \\
DX
\end{array}
$$

Thm: For sbs S, PS, $D_{\leq 1}S$, $D_{<\infty}S \in$ Sbs

and agree with their counterparts on Meas.

Open: Is there a conterpart to D in Meas?

(Hypothesis: no)

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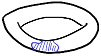
- Borel sets 
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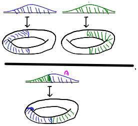
  def., constructions, 

  partiality, refinement $\to, \Rightarrow, \coprod, \prod$
- Measures & integration $\delta, \int, \oint, D, \otimes, P$
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Sneak Peek

Tomorrow will deal with more higher-order spaces:

Random Variable Spaces

- $\mathbb{R}^{\Omega}$, $\bar{\mathbb{R}}^{\Omega}$, W^{Ω} etc.
- $\mathcal{L}^{p}_{(\Omega,\mu)} := \left\{ \varphi : \Omega \to \mathbb{R} \;\middle|\; \int \mu \, |\varphi|^{p} < \infty \right\}$