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Subtitles for

DEUX OU TROIS CHOSES QUE JE SAIS D'ELLE

(LA LOGIQUE)

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I — The end of systems

	Analytic	Synthetic
Explicit	Constat	Usine
Implicit	Performance	Usage

LL : central role of usine (proof-nets).

	Type – free	Typed
Explicit	Data	Cut – free
Implicit	Programs	Deductive

In technical terms.

1 – BHK, HERBRAND, GENTZEN

- **Logical system :** ***oxymoron,*** **logic should not be sectarian.**

  LL : not yet another system, since *system-free!*

- ***Gentzen*** **subformula property.**

  Proof of A : involves the sole A, not full system.

- ***Herbrand*'s theorem, prefiguration of *proof-nets* (usine).**

  $\forall x \exists y A[x, y] \neq \exists y \forall x A[x, y]$ **: explained by *tests*.**

  $A[x, t]$ **: make** $x =: f(y)$ **in case** $\exists\forall$.

- ***BHK*, proofs as functions, meaning is *use* (usage), not checkable (usine).**

  Proof of $A \Rightarrow B$: function from proofs of A to proofs of B.

  Performative : from constat to constat (cut-elimination).

- ***Open*** **architecture ; independent parts compatible.**

  Usage : Curry-like (existentialist, with cut).

  Usine : Church-like (not quite essentialist, cut-free).

  Adequation : external, proof of normalisation.

2 – EQUALITY

- « **Leibniz equality** » : any property of a is a property of b.

$$a = b := \forall X(X(a) \multimap X(b))$$

- *Quid* of « left of $=$ » ?

  Shocking : not *semantic*, does not preserve... equality!

  Systems : avoid aporia by selecting properties *compatible* with equality.

- Transitivity $\vdash X(a) \otimes \sim X(b), X(b) \otimes \sim X(c), \sim X(a) \parr X(c)$

  Semantic mistake : remove all $X(\cdot)$ and everything works !

  Rewrites as : $\vdash a \otimes \sim b, b \otimes \sim c, \sim a \parr c$

- *Individuals as propositions* suppose *linear* maintenance.

  Intuitionistic : $\neg\neg((a \equiv b) \vee (b \equiv c) \vee (c \equiv a))$

  Equality : $a \multimap b \,\&\, b \multimap a$: $X(x) := x, X(x) := \sim x$.

- Leibniz : logical *epicycle.*

  Metaphysics : individuals and their « properties » sort of viruses.

- Non linearity : need system to avoid *miscegenation*.

3 – ADDITIVES

- **Second order** $A \oplus B := \forall X((A \multimap X) \Rightarrow ((B \multimap X) \Rightarrow X))$

$$\frac{\begin{array}{c}\vdots\\ A\end{array}}{A \oplus B}\,(l \oplus I) \qquad \frac{\begin{array}{c}\vdots\\ B\end{array}}{A \oplus B}\,(r \oplus I) \qquad \frac{\begin{array}{c}\vdots\\ A \oplus B\end{array} \quad \begin{array}{c}[A]\\ \vdots\\ C'\end{array} \quad \begin{array}{c}[B]\\ \vdots\\ C''\end{array}}{C}\,(\oplus E)$$

- **No subformula property, *extraneous* C may hide a cut.**

  Prawitz : *commutative* conversions against the performative paradigm.

- **LL solution (new !) : delogicalise C', C'', C, into plain *locations* a, b, c.**

  Switching L/R : connect c with a *or* b.

  Hidden cuts : normalised during « compilation » (correctness).

  Extra locations : a, b, c part of formula $A \oplus B$.

  Rule $(l \oplus I)$: proof of A and link $[\![\, a, c \,]\!]$

  Rule $(\&)$: proof of A, a, proof of B, b and link(s) $[\![\, c, \Gamma \,]\!]$

4 – EMPTY TYPES

- **Nightmare of *realisability* (cheap BHK), e.g., $\neg A \vee \neg\neg A$.**

  Invisible proofs : excluded since *false* but logically correct.

  Example : switchings of $\sim A$.

  Question : difference between *legit proof* and plain counter-switching.

- **Second order : synthetic (subjective) features due to *existence.***

  Proof of $\exists X\, A$: involves switchings of the witness.

- **Euler-Poincaré invariant for $[\![t_1, \ldots, t_k, u_1, \ldots, u_\ell]\!]$:**

  Objective : if $\ell = 0$, then $|[\![t_1, \ldots, t_k]\!]| = 2 - k$.

  Subjective : if $\ell \neq 0$, then $|[\![t_1, \ldots, t_k, u_1, \ldots, u_\ell]\!]| = -k$.

  Visibility : $\mathcal{P}$ *true* when $|\mathcal{P}| \geq 0$.

- **Truth preserved by rules, *including cut,* and normalisation.**

  Derealistic : applies to proofs, not to formulas ; no semantics !

  Hidden proofs : essential to architecture like hidden files (`.emacs`).

  Absurdity : $0 := !(フ \parr ヲ) \otimes ヲ$. Not empty.

5 – ARITHMETIC

- **Multiplicative constants** $1, \bot$ **essentialist fantasy of** ***category theory.***

  Don't work : no correctness criterion, hence ***impossible.***

  Actual constants : フ, ヲ **(katakanas** ***fu*** **et** ***wo*****).**

  Atoms フ **objective,** ヲ **subjective. Self-dual** フ $=\sim$フ, ヲ $=\sim$ヲ.

  Absurdity : $\mathbf{0} := !($フ $⅋$ ヲ$) \otimes$ ヲ.

- **Desaxiomatisation of Peano** ***arithmetic.***

  Second order : multiplication and recurrence à la ***Dedekind***

  $\mathrm{x} \in \mathbb{N} := \forall \boldsymbol{X}\ (\forall \mathrm{y}\ (\boldsymbol{X}(\mathrm{y}) \multimap \boldsymbol{X}(\mathrm{y} + \bar{1})) \Rightarrow (\boldsymbol{X}(\bar{0}) \multimap \boldsymbol{X}(\mathrm{x})))$

  Individuals : $\bar{1} :=$ フ$, \quad \mathrm{x} + \mathrm{y} := \mathrm{x} \otimes \mathrm{y}, \quad \mathrm{x} - \mathrm{y} := \mathrm{y} \multimap \mathrm{x}.$

  Axioms 3 and 4 : e.g., $\mathrm{x} + \bar{1} \equiv \mathrm{y} + \bar{1} \multimap \mathrm{x} \equiv \mathrm{y}$ **prouvables.**

- **Works thanks to** ***derealistic*** **truth.**

  True and false : $\bar{1} :=$ フ **true,** $\overline{-1} :=$ フ $⅋$ フ $⅋$ フ **false...**

  But : $\bar{1} + \overline{-1} \equiv \bar{0}$ $(\equiv$ フ $⅋$ フ$)$ **true.**

  $\bar{1} \otimes \overline{-1}$ **true : contradicts tarskian pleonasm and** ***any*** **truth table.**

II — LOGIC AND OPERATORS

The actual analytics.

6 – GoI

- **Analytic : beyond discussion *(computation).* Quid of analytic *space ?***

  Pure λ-calculus robust and *archaic :* no linearity.

  Operator algebras : C^*, von Neumann.

- **Upper part of *multiplicative* proof-net seen as :**

  Permutation : leads to *Geometry of Interaction.*

  Partition : adapted to « Schrödinger's cut. »

- **GoI : cut-free proof as *partial symmetry,* $u^3 = u, u = u^*$**

  Application of « function » F to « argument » a (cut-elimination).

  $[F]a := (1 - a^2)(1 - Fa)^{-1}F(1 - a^2)$

  Provided $1 - Fa$ invertible.

- **The ultimate BHK. But takes care of the sole *usage* (proofs as functions).**

  Termination : Fa nilpotent, e.g., $(Fa)^{N+1} = 0$:

  $(1 - a^2)(F + FaF + FaFaF + \ldots + F(aF)^N)(1 - a^2)$

  Algebra : finitistic (prestellar), to be defined.

7 – THE ALGEBRA $\mathbb{U}$

- **Fixed functional language ; irrelevant provided contains $g(\cdot,\cdot)$ and a.**

  Monomials : tu^* **with *same* bound variables.**

  Examples : $f(x)g(x,x)^*$ **but** $f(x)g(x,y)^*$ **excluded.**

  Composition : $tu^* \cdot vw^* := t\theta w\theta^*$ **if** u, v ***unifiable*** **with *mgu*** θ.

  Otherwise $tu^* \cdot vw^* := 0$.

  Example : $xx^* \cdot tu^* = tu^*$.

  Involution : $(tu^*)^* := ut^*$ **; *contravariant* compositionwise.**

- **Implemented on *Hilbert space*** ℓ^2 **spanned by closed terms.**

  $tu^*[a] := b$ **when** $tu^* \cdot aa_0^* := ba_0^*$ **; works since *same* variables.**

  $tu^*[a] := 0$ **when** $tu^* \cdot aa_0^* := 0$.

  Involution implemented as *adjoint.* Indeed $t, u : \ell^2 \otimes \ldots \otimes \ell^2 \mapsto \ell^2$.

- **Algebra $\mathbb{U}$ spanned by the** tu^* **admits a stellar norm** $\|uu^*\| = \|u\|^2$**.**

  Hence a *greatest* one ; the one to select in case.

- **Keep $\mathbb{U}$ *algebraic* including complex coefficients.**

8 – Normalisation, revisited

- **Suggested by quantum physics (Schrödinger's cut).**
  Projections : $u = u^* = u^2$ **(correspond to Danos-Regnier *partitions*).**
  Implemented as closed *subspace* $|u|$ **of Hilbert space** ℓ^2**.**

- ***Replace* partial symmetries, typically :**
  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = f(x)g(x)^* + g(x)f(x)^*$ **with :**
  $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \sqrt{2}/2(f(x)+g(x))\sqrt{2}/2(f(x)^*+g(x)^*)$.

- ***Intersection*** $u \cap v$ **as « limit » of decreasing sequence**
  $u \geq uvu \geq uvuvu \geq \ldots \geq (uv)^n u \geq \ldots$
  Provided $\mathrm{Sp}(uv) \subset \{\lambda_1, \ldots, \lambda_n, 1\} \quad (0 \leq \lambda_i < 1)$
  $uv \cdot (1-\lambda_1)^{-1}(uv-\lambda_1) \cdot \ldots \cdot (1-\lambda_n)^{-1}(uv-\lambda_n) \cdot u$
  Dually : $u \uplus v := 1-((1-u) \cap (1-v))$ **closed linear span of *union*.**

- **Application of « function »** F **to « argument »** a.
  $[F]a := (F \uplus a) \cap (1-a)$, **provided** $F \cap a = 0$.
  Graph-like : $x \in |a|, (-x) \oplus y \in |F| \mapsto y \in [F]a$.

III — ENVOI

What logic is about.

9 – A PROMETHEAN PROJECT

- ***Legitimate*** **doubts about :**

  Principles : church, politics, axiomatics (= army).

  Realism : semantics is *reasoning* about a « reality » we are part of.

- **Pure reason : what can we *sure* of ?**

  Aristotle : syllogistics refers to itself.

- **Mistreated by scientism and obsolete *foundational* projects.**

  Axiomatic realism : language refers to reality $(\forall \Rightarrow \exists)$.

  Systems : contingent truths ; same reality, different spectacles.

- **Linearity.**

  Not yet another system : linear implies system-free.

- ***Open*** **architecture ; independent parts compatible.**

  Usage : Curry-like (existentialist, with cut).

  Usine : Church-like (not quite essentialist, cut-free).

  Adequation : external, proof of normalisation. Forget foundations !